T Dwight Townsend

AN

ELEMENTARY TREATISE

ON

ALGEBRA,

IN WHICH

THE PRINCIPLES OF THE SCIENCE

ARE

FAMILIARLY EXPLAINED, AND ILLUSTRATED BY NUMEROUS EXAMPLES.

Designed for the Use of Schools.

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PREFACE.

ALGEBRA is justly considered one of the principal foundations of all sound mathematical knowledge. Since the investigations of modern geometers have given to analytical investigations that predominance which they now hold over the synthetical methods adopted by ancient mathematicians, its importance has proportionably increased. Every person, therefore, who wishes to obtain a thorough knowledge of the higher mathematics, must commence by studying and fully mastering the principles of Algebra.

It is not to such persons alone that it is important. The habits of investigation to which it leads; the powers of analysis which it confers; and its general application to the solution of problems, which are frequently presented to every person who lays any claim to a liberal education, make it an important, if not an essential branch of education.

The object of this treatise is to present the science in a manner sufficiently simple to enable all to understand it, and yet so comprehensive as to embrace nearly every thing that it is necessary for the student to learn, as a preparation for his future studies. The first part of the work, which includes Quadratic Equations, will be found to be more full than common, particularly on the subject of pure equations. It is believed to contain all that is required, for one who desires to obtain a knowledge of the more elementary parts of Algebra. The remaining chapters contain the theory of Equations, Series, Logarithms, Indeterminate and Diophantine Analysis.

Most teachers have found that children commencing the study of Algebra are frequently at a loss to understand the nature of the operations they are required to perform. The addition and subtraction of letters seem to them foolishness. Some preliminary exercise is necessary to enable them to perceive the utility of their labours. It is hoped the preliminary chapter in this work will do something towards removing this inconvenience. The questions selected are so simple that no child who is prepared to commence the study of Algebra will find much difficulty in performing them; no operations being necessary but such as the method of instruction universally practised by all thorough teachers of arithmetic will have rendered familiar. In solving the various questions that are found in this chapter, the student can hardly fail to become familiar with the meaning and use of the symbols; and thus be prepared to enter upon the subsequent portions of the work, without that embarrassment to which allusion has been made. It is earnestly recommended that he be made fully acquainted with this chapter before he is allowed to proceed.

Considerable care has been taken to make the explanation of the various rules concise, yet clear. The attention of teachers is particularly called to the remarks on the absolute negative quantity, art. 11 and 12; in which an attempt has been made to relieve the pupil from a cause of embarrassment to which he is generally subjected when commencing his labours. The demonstration of the rule for *signs* in multiplication and division, has no claim to novelty. Notwithstanding its importance, it is often omitted in elementary algebras. The omission of such demonstrations will at once be seen to be objectionable, when it is remembered that children are taught Algebra for the purpose of putting in their hands an instrument by which they may remove difficulties they meet with elsewhere. Such explanations should never be passed over without being understood; an opposite practice leads to loose habits of study, which often lay the foundation for much future difficulty, and deprive the pupil of the satisfaction which he would feel from the consciousness that every thing in the work he had studied had become his own.

The method employed, art. 17, in explaining the force of the index, was generally used by ancient authors it has been too much neglected in modern treatises. It will be found to give more precise notions respecting the exponent than can be obtained in any other way.

Throughout the first part, numerous examples have been given, sufficient, it is believed, to familiarize the student with all the methods of solution employed.

In the Second Part, the theory of equations has been much more fully developed than in any elementary treatise with which the author is acquainted. Care has been taken to preserve perfect rigour in the demonstrations. Some of these will be found to be very concise. The beautiful theorem of M. Sturm, for which he obtained the mathematical prize from the French Academy, has been developed at some length; as well as the compendious method of Horner for approximating to the values of the roots of an equation. The chapter on the Summation of Series has been principally taken from Young's Algebra; that on Binomial Equations from a treatise on the theory of equations, by the same author. For the theory of Diophantine Analysis, the author is principally indebted to the admirable treatise on algebra by Euler.

In the preparation of the work, most of the treatises on the subject in common use have been consulted, more, however, for the purpose of discovering what had been done, than from an expectation of deriving much direct assistance from them. For the greater part of the theory, the author is only so far indebted to books as they have enabled him to store his own mind with knowledge on the subject. In selecting examples, however, he has made free use of all the treatises in his possession. A considerable number have been taken from "Bland's Algebraical Problems."

In conclusion, the author would remind those who may be disposed to use the work, that in a treatise of this kind much that is new could not be expected. Most that can be done is to simplify the arrangement, and render the demonstrations more clear and precise. If this result has been obtained, and an important branch of science has thus been made more accessible, one great point has been gained. With these remarks the author leaves the work to the judgment of an enlightened public.

Philadelphia, 5th month, 1846.

CONTENTS.

SYNOPSIS OF THE DEFINITIONS.

∴ This sign is used for *therefore.*

∝ Signifies *infinity.*

∽ Indicates the difference between two quantities between which it is placed.

Affirmative quantities are those affected with the sign $+$. A quantity without a sign is always considered *affirmative.* (Art. 10.)

Negative quantities are those affected with the sign $-$. (Art. 10.)

Monomial quantities are those whose parts are not separated by the sign $+$ or $-$; thus a, $5\,b^2$, and $6\,bc^2$ are *monomials.*

Binomial quantities are such as consist of two monomials, connected by the sign $+$ or $-$; thus $a+b$ and $3\,a^2-5\,bc$ are *binomials.*

The monomials which form a binomial are called *Terms.*

Polynomial quantities consist of more than two terms; thus $4\,a-5\,b^2x+c^2$ is a polynomial.

Coefficients are numbers joined to any quantity to indicate how often it is considered as being repeated; thus $3\,a$ is the *coefficient* of x in the expression $3\,ax$. (Art. 16.)

Index or *Exponent*, is a number or symbol placed over an expression to indicate some power or root; thus 2 is the exponent of a^2; $\frac{2}{3}$ of $(a+b)^{\frac{2}{3}}$. (Art. 18 and 89.)

Homogeneous quantities are those which contain the same number of factors; thus x^3y^2 and a^2xy^2 are homogeneous, each containing 5 factors. (Art. 25.)

Ratio is the relation which one quantity bears to another in magnitude, and is expressed by dividing the second by the first; thus the ratio of 4 to 5 is $\frac{5}{4}$.

Equation is an expression of equality between two expressions; the two expressions considered equal being called the *members* or *sides* of the equation. (Art. 6.)

ALGEBRA.

CHAPTER I.

DEFINITIONS AND PRELIMINARY EXAMPLES.

ART. 1.—Algebra is the science of computing by arbitrary characters. By it we are also able to trace many abstract relations of numbers, which can not be done by common Arithmetic.

The quantities employed in algebraic calculations are represented by symbols; for which purpose the letters of the alphabet are generally employed.

The different operations upon these quantities are indicated by signs, with most of which the student has become familiar in Arithmetic. Thus

2. To represent addition, we make use of the sign $+$, (*plus*) or more. $a + b$, which is read a plus b, signifying that the quantity represented by b is added to that represented by a.

3. $-$ (*minus*,) or less, placed between two letters, indicates that the quantity represented by the latter of these letters is to be subtracted from that represented by the former. Thus $a - b$ is read a minus b, and signifies the remainder left by taking b from a.

4. $\times$ is used to indicate the product of the quantities between which it is placed. $a \times b$ is read a multiplied by b.

Multiplication is often expressed by placing a dot between the factors, or by simply writing them as in a word. Thus, $a. b$, or ab, indicates the product of the factors a and b, and is consequently identical with $a \times b$. Similarly, $3\,a$ and $2\,x$ indicate respectively three times the quantity represented by a, and twice that represented by x.

5. The division of quantities is expressed by writing the divisor after the dividend, and separating them by the sign ÷; or by placing them as a vulgar fraction, the divisor of course being the denominator. Thus, $a \div b$, or $\frac{a}{b}$ signifies the quotient arising from dividing a by b, and is read a divided by b.

6. The sign =, (*equal*,) expresses the equality of the quantities between which it is placed.

An expression of equality is called an *equation;* the quantities represented as equal being called the *members* or *sides* of the *equation*.

7. To exhibit the conciseness which results from the use of these symbols, we shall employ them in the solution of the following problem.

It is required to divide $1000 amongst three persons, A, B, and C, so that B may have $50 less than A, and C $125 more than B.

By the use of ordinary language it may be solved thus:	Algebraically thus: Let x represent B's share.
A has $50 more than B.	Then A's is $x + 50$.
C has $125 more than B.	C's is $x + 125$.
Therefore the three will have $175 more than three times the share of B.	Therefore the three will be $3x + 175$.
Consequently three times the share of B, and $175, must make the sum to be divided, or $1000.	We will consequently have $3x + 175 = 1000$,
Three times the share of B is, therefore equal to $1000 diminished by $175, or to $825.	or $3x = 1000 - 175 = 825$,
Hence the share of B is equal to $\frac{\$825}{3}$ or $275,	and $x = 275 =$ B's.
and A's equals 325.	$x + 50 = 325 =$ A's.
C's " 400.	$x + 125 = 400 =$ C's.

In the above example, the solution by the use of ordinary language was easy; in many cases, however, this is difficult even when it may be performed with great readiness by the use of algebraical symbols.

The following simple examples will enable the pupil fully to understand the advantage, which the use of the symbols above explained possesses; and will render him familiar with their application.

Ex. 1.—Required to divide a line of 125 yards in length into three parts, such that the middle may be one-third as long as one of the extremes, and ten yds shorter than the other.

Here, if we represent the middle portion by x, the extremes will be $3x$, and $x + 10$, respectively. The whole line will therefore be $5x + 10$. Consequently, we have the equation $5x + 10 = 125$

and $5x = 125 - 10 = 115$. (A)

whence $x = 23$ the middle portion.

$3x = 69$
and $x + 10 = 33$ } the extreme portions.

Ex. 2.—A post is half its length in the mud, fifteen feet in the water, and one-third of its length above water. Required its length.

Let x represent the length, then the separate portions will be

$\frac{x}{2}$ in the mud.

$\frac{x}{3}$ in the air.

15 feet in the water.

Consequently $\frac{x}{2} + \frac{x}{3} + 15$ is the length of the post.

Hence $x = \frac{x}{2} + \frac{x}{3} + 15$.

To avoid the embarrassment arising from the fractions we multiply the several terms of the expression by 6, which gives $6x = 3x + 2x + 90 = 5x + 90$

Subtracting $5x$ from each member, we have

$6x - 5x$, or $x = 90$, the length required. (B)

8. There are some important remarks to be made on the processes employed in the preceding solutions.

1st. *Any quantity may be transposed from one member of an equation to the other, if we change its sign.*

This is exemplified in the equation marked (A) in the first example; 10 having been there transposed from the left-hand member to the right, its sign being at the same time changed from + to —: and likewise in the equation (B), of the second example, where $5x$ has been taken from the right to the left-hand member, its sign being changed.

The correctness of this operation is manifest from the principle, that equals, increased or diminished by equals, must still be equal. Thus, in the first case above alluded to, the left-hand member is $5x + 10$. If, then, we change it to $5x$, we diminish its value by 10; and, consequently, the right-hand member, 125, must likewise be diminished by 10; which changes it into $125 - 10 = 115$; so that the equation will read $5x = 125 - 10 = 115$.

Had the original equation been $5x - 10 = 125$, it is evident that the left-hand member is 10 less than $5x$, and must, therefore, be increased by 10 to make it $5x$. Increasing the other member by the same number we should have $5x = 125 + 10 = 135$.

9. *An equation may be cleared of fractions by multiplying all its terms by the least or any other common multiple of the denominators.*

The reason of this is plain.

Ex. 3. A father in his will directed his property to be divided amongst his daughter and two sons, in the following proportions, viz.: the elder son was to have one-half the estate, less $13000; the second son was to have one-third, less $2000; and the daughter was to receive one-fourth and $3500. He likewise directed the remainder, which was ascertained to be $6000, to be given to the "Pennsylvania Asylum for the Blind." Required the estate and the shares of the children?

Here, if the whole estate be represented by x,

the elder son's share will be - - $\frac{1}{2}x - 13000$

Younger's - - - - - - - $\frac{1}{3}x - 2000$

Daughter's - - - - - - $\frac{1}{4}x + 3500$

Consequently

$$\tfrac{1}{2}x - 13000 + \tfrac{1}{3}x - 2000 + \tfrac{1}{4}x + 3500 + 6000 = x$$

Clearing of fractions, by multiplying by 12; we have

$$6x - 156000 + 4x - 24000 + 3x + 42000 + 72000 = 12x$$

or, transposing,

$$6x + 4x + 3x - 12x = 156000 + 24000 - 42000 - 72000$$

that is,

$$13x - 12x = 180000 - 42000 - 72000 = 138000 - 72000 = 66000$$

or, $\quad x = 66000$ the whole estate,

and $\left.\begin{aligned}\tfrac{1}{3}x - 13000 &= 20000\\ \tfrac{1}{3}x - 2000 &= 20000\\ \tfrac{1}{4}x + 3500 &= 20000\end{aligned}\right\}$ the children's shares.

Ex. 4. What number is that, to the double of which if 18 be added the sum will be 96?

Here the equation will evidently be $2x + 18 = 96$.

Ans. 39.

Ex. 5. What number is that, from five times which if we subtract 24 the remainder will be 196?

Or, $\quad 5x - 24 = 196$ *Ans.* 44.

Ex. 6. In a certain school, if the number of boys be doubled, and then increased by 25, the result will be 367. How many are there? *Ans.* 171.

Ex. 7. What number is that whose double exceeds its half by 78? *Ans.* 52.

Ex. 8. A number increased by its half, then by its third, and afterwards diminished by 56, makes 164. What is that number? *Ans.* 120.

Ex. 9. In a certain orchard, one-half the trees bear apples, one-fourth bear plums, one-fifth peaches, and twenty bear cherries. How many in all? *Ans.* 400.

Ex. 10. What number is that, which being increased by 75, the result shall be four times the original number?

Ans. 25.

Ex. 11. A and B set out from Philadelphia towards Baltimore. A has 3 hours the start, and travels 5 miles per hour. B travels 7 miles per hour: how long will he be in overtaking A, and how far will he travel before that occurs?

Ans. Time, $7\tfrac{1}{2}$ hours; distance, $52\tfrac{1}{2}$ miles.

Ex. 12. What number is that whose fourth part exceeds its fifth part by 25? *Ans.* 500.

Ex. 13. A gentleman purchased a horse, a chaise, and harness, for $1000. The horse cost four times as much as the harness, and the chaise three times as much as both. Required the price of each.

Ans. Harness $50; horse $200, and chaise $750.

Ex. 14. The head of a fish is 11 inches long, its tail is as long as its head and half its body; and its body is as long as its head and tail. What is the length? *Ans.* 7 ft. 4 in.

Ex. 15. One-fourth of the contents of a cask leaked out, ten gallons and a half were afterwards drawn out, after which the cask was found to be two-thirds full. What was the whole content of the cask? *Ans.* 126 gallons.

Ex. 16. One-fifth of the boys in a school are studying arithmetic, one-third algebra, one-fourth geometry, and 13 are studying surveying. What is the whole number?

Ans. 60.

Ex. 17. A criminal having escaped, travels 16 hours per day, at the rate of 3 miles per hour; after three days his route is discovered, and an officer, starting in pursuit, travels 12 hours per day at the rate of 5 miles per hour, how long before he overtakes the fugitive, and how far will they have gone? *Ans.* 12 days, and 720 miles.

Ex. 18. An estate of $39,000 is to be divided amongst A, B and C, in the following manner: C's share is to be one-third of A's, and B's is to be equal to C's and half of A's. What is the share of each?

Ans. A, 18,000; B, 15,000; C, 6000.

Ex. 19. Bought a piece of cloth which proved to be only $\frac{7}{8}$ as long as it was marked, nevertheless, by selling it at $6·00 per yard, I received as much as it cost. What was the cost per yard? *Ans.* $5·25.

Ex. 20. A servant was hired at $62\frac{1}{2}$ cents per day for a year, consisting of 313 working days, on condition that he should be charged $37\frac{1}{2}$ cents for his board every day he was idle. On settlement, it was found there was \$145·$62\frac{1}{2}$ due him. How many days was he idle? *Ans.* 50.

Ex. 21. A and B commence trade with the same capital. The first year A gains \$5000, and B loses one-fourth of his stock. When A's money is treble B's. What was their capital? *Ans.* \$4000.

Ex. 22. Three men purchased a ship. A paid $\frac{3}{20}$ths, B, $\frac{5}{12}$ths, and C the remainder, which was \$7800. What was the whole cost? *Ans.* \$18,000.

Ex. 23. A can do a piece of work in 12 days, but wishing to have it finished in less time, he hires B, and the two perform it in 7 days. In what time could B alone have done it? *Ans.* $16\frac{4}{5}$th days.

Ex. 24. A woman purchased some eggs at 10 cts. per doz., and twice as many at 9 cts. per doz. She sold them at 12 cts. per doz., and thereby gained 96 cents. How many did she purchase altogether? *Ans.* 36 dozen.

Ex. 25. The sum of two numbers is 25 and their difference is 12. What are the numbers? *Ans.* $18\frac{1}{2}$ and $6\frac{1}{2}$.

Ex. 26. There are three numbers whose common difference is 4 and sum 48. What are the numbers?
Ans. 12, 16 and 20.

Ex. 27. A, B and C can perform a piece of work in 5 days. A alone can do it in 12 days, and B in 15. In what time could C accomplish it? *Ans.* 20 days.

Ex. 28. Required to divide a line of 99 inches into three such parts, that $\frac{1}{2}$ the first, $\frac{1}{3}$ the second, and $\frac{1}{4}$ the third shall be equal. What are the parts?
Ans. 22, 33, and 44 inches.

CHAPTER II.

ON THE PRELIMINARY RULES.

SECTION I.

On the Addition of Algebraic Quantities.

10. BY the addition of algebraic quantities, is understood the collecting them together; performing with each the operation indicated by its sign. Thus when we collect the quantities in the expression $6+5-3+2$, we find the result to be 10. This operation is considered to be one of addition, although one of the processes is really a subtraction.

In regard to the addition of *positive* numbers, (those affected by the sign $+$,) no difficulty can arise, since the operation is manifestly performed in the same manner as in arithmetic. Thus $6+4+3=13$, and $6x+4x+3x=13x$, as much as 6 apples + 4 apples + 3 apples = 13 apples.

If *dissimilar* quantities are required to be added, we can only do it symbolically. Thus, if Thomas received from one man \$5, from a second 3 yards of cloth, from a third 2 yards of cloth, and from a fourth \$12. He receives altogether \$17 + 5 yards. So that \$5 + 3 yds + 2 yds + \$12 = \$17 + 5 yds.

So also $9a+5x+3x+2a=11a+8x$.

If any of the quantities are *negative* (that is, are affected with the sign $-$) they must be *subtracted* from the sum of the like *positive* terms. Thus let the value of the expression $75-37-24$ be required. This expression evidently means 75 diminished by 37, and the result diminished by 24. We therefore have

$$75-37-24=38-24=14.$$

Now it must be evident that diminishing a number successively by two others, is equivalent to diminishing it by their sum. Consequently

$$75 - 37 - 24 = 75 - 61 = 14 \text{ as before.}$$

We here see that to collect two negative quantities we add them and prefix the common sign —.

Let now the result of the following operations be required, viz.,

$$7 + 5 - 3 + 8 - 6 - 4 + 12.$$

It may be reduced in the following manner:

$$7 + 5 - 3 + 8 - 6 - 4 + 12 = 12 - 3 + 8 - 6 - 4 + 12$$
$$= 9 + 8 - 6 - 4 + 12$$
$$= 17 - 6 - 4 + 12 = 11 - 4 + 12 = 7 + 12 = 19.$$

This process is, however, tedious, and may be abbreviated by observing, that *in general* it can make no difference in the final result, whether we collect the quantities in the order in which they were written or in any other that may be more convenient. The above quantity is therefore the same as $7 + 5 + 8 + 12 - 3 - 6 - 4 = 32 - 13 = 19$.

When the quantities are dissimilar, they of course can only be so far collected as to include in separate amounts those of the same kind.

11. Again, let it be required to collect the following quantities:

$$3 + 4 - 10 + 12,$$

We may proceed thus:

$$3 + 4 - 10 + 12 = 7 - 10 + 12$$

but here we are met by a difficulty, since the next operation, which requires us to subtract 10 from 7, is manifestly impossible.

Such cases generally indicate some absurdity in the conditions, as will be seen by the following example.

A snail commenced climbing a pole. The first hour he ascended 3 feet, the next 4 feet, the third he descended 10 feet, and again ascended 12 feet the fourth hour. What is his elevation at the termination of the four hours?

This problem will give the expression above, viz.,

$$3 + 4 - 10 + 12,$$

and is absurd, since, when he had ascended but 7 feet, it was impossible to descend 10 feet.

In such cases it is usual to deduct the positive from the negative quantity, and prefix the sign — to the remainder. Thus $7 - 10 = -3$; by this means reducing the absurdity to another form.

12. We must not suppose, however, that negative results always indicate impossible conditions. The following is an instance in which no absurdity is implied.

A gentleman started out to collect some debts. He first obtained from A \$100, then from B \$300; after which he paid C \$700, and finally received from D \$150. What was the result of his day's operations?

The formula is evidently

$$\begin{aligned} & 100 + 300 - 700 + 150 \\ = & \quad 400 - 700 - 150 \\ = & \quad -300 + 150 \\ = & \quad -150 \end{aligned}$$

We have in this case the same difficulty as before; but there is no absurdity, unless the gentleman had no funds, on which he could draw to pay the \$700. In case there were such funds, they would be diminished \$300 by this payment, and \$150 by the whole day's operations.

So in the former case, had we supposed the snail to set out at a point more than 3 feet high, the absurdity would cease to be other than apparent; the result $7 - 10 = -3$ merely showing that at the end of the third hour he had arrived at a point 3 feet lower than that from which he had started. The result $-3 + 12 = 9$ indicates a final progress of 9 feet. So that, had his original elevation been 5 feet, he would have arrived at the height of 14 feet at the end of the four hours.

Let it now be required to add the quantities

$6a - 4b - 3c$, $7a - 2b + 4c$ and $6c + 2b - 3a$.

The result may be written

$$\begin{aligned} 6a + 7a - 3a - 4b - 2b + 2b - 3c + 4c + 6c \\ = 10a - 4b + 7c. \end{aligned}$$

In performing addition, therefore, *collect the similar quantities from all the expressions to be added, operating with each as indicated by its sign; that is, collect all the positive quantities of the same kind into one sum, and the negative into another: take the difference of the results which must be affected with the sign of the greater.*

EXAMPLES.

Ex. 1.—Add

$$\begin{array}{rrr} 5a+4b & - & 3cx \\ 7a-3b & + & 2cx \\ -3a-7b & - & 3cx \\ 9a+5b & + & 12cx \\ \hline 18a-b & + & 8cx \end{array}$$

2.

$$\begin{array}{llll} 6a & +\ 5y & -6bc \\ 7bc & +\ 4a & -6y \\ 12y & +\ 7a & -3bc \\ 15y & +\ 2bc & -4a \\ \hline 13a & +\ 26y & \end{array}$$

In these examples the positive quantities are collected into one amount and the negative into another, and the difference taken, which is set down with the sign of the greater sum. Thus, $5a+7a+9a=21a$, and $21a-3a=18a$; again, $4b+5b=9b$, $3b+7b=10b$, and $9b-10b=-b$, so of the rest. The positive and negative amounts in the case of bc in the second example are equal, and therefore the remainder is nothing.

Ex. 3. Required the sum of the following quantities, viz.: $3a-2b+4cx$, $7cx-3b+8a$, $-9a+3cx-5b$ and $2a-3cx+4b$.

Ex. 4. Required the sum of $3ax-4bc+12cx$, $7cx-5ax+14bc$, $8ax-12bc+3cx$, and $2bc-6ax+8cx$.

Ex. 5. Add $3ay+4bx-5ac$, $7bx-3ac+2ay$, $8ac-7ay+2bx$, and $9ac-3bx+7ay$.

Ex. 6. Add $3abc-4ac-\frac{1}{2}bc$, $3ac+\frac{3}{2}abc-7ac$, and $9\frac{1}{2}ac+abc-bc$.

Ex. 7. Add $ax-4ab+bd$, $3bd-2ax+ab$, $7ab-2ax-bd$, and $5ab-3ax+12bd$.

Ex. 8. Add $3abd+4abx-5cx$, $8cx-11abx+12abd$, $9abx-12cx+8abd$, and $7cx-15abx+3abd$.

SECTION II.

Subtraction.

13. Subtraction being the reverse of addition, it is evident that we must apply every term in the subtrahend, with the opposite sign from what we should were the quantities to be added. *Therefore, to subtract one algebraic expression from another we must change all the signs in the subtrahend, and then proceed as in addition.*

This will be made plain by the following examples, viz.:

$$\left.\begin{array}{ll}\text{add} & 6a+5b\\ \text{and} & 3a-2b\\ \text{sum} & \overline{9a+3b}\end{array}\right\}\text{consequently}\left\{\begin{array}{ll} & 9a+3b\\ \text{dim'd by} & 3a-2b\\ \text{equals} & \overline{6a+5b}\end{array}\right.$$

Now this latter result would equally be obtained by adding

$$\begin{array}{rr} & 9a+3b\\ \text{and} & -3a+2b\\ \text{since the result is} & \overline{6a+5b}\end{array}$$

and this operation is evidently in accordance with the rule.

14. The reason of the above rule may perhaps be made more clear by the following illustrations.

If we diminish any number, as 50, by the sum of any numbers, say 15, 6, and 9; it can evidently make no difference whether we diminish it separately by the numbers themselves or first find their sum, and then subtract this. The former of these operations leads to the formula.

$$50-15-6-9$$

in which the subtracting terms are set down with the sign —.

Let it now be required to ascertain the remainder arising from subtracting $30-10$ from 50.

If we diminish 50 by 30, we have $50-30=20$ for the remainder. This result is evidently too small, since the subtrahend was too great by 10. To obtain the true remainder we must evidently increase that so obtained by 10; so that we shall have for the final result,

$$50-30+10 \text{ or } 20+10=30.$$

The student may satisfy himself of the correctness of this process, by first reducing $30 - 10$, and then subtracting the result. Thus, $30 - 10 = 20$, and $50 - 20 = 30$, as before.

15. To generalize the above reasoning we may proceed thus. Let it be required to subtract $a - b$ from x. Now it is evident that the quantity $a - b$ is b less than a. If, therefore, we subtract a from x, the remainder $x - a$, will be too small by b, and will therefore require to be increased by b. The true result is, therefore,

$$x - a + b$$

in which each term of the subtrahend is applied with a contrary sign.

EXAMPLES.

Ex. 1. From $7ax - 3bc + 8by$
take $4ax + 3bc - 2by$

$$3ax - 6bc + 10by.$$

Ex. 2. From $8x - 4bx + 9bc$
take $3bx - 7bc + 2x$

$$6x - 7bx + 16bc.$$

Ex. 3. From $8ax - 3by + 4dx$ take $3dx - 5ax + 3by$.

Ex. 4. From $9bcx + 7aby - 4bx$ take $3bcx + 2aby - 4bx$.

Ex. 5. From $3bx - 4acy + bxy$ take $2acy - 5bx - bc$.

Ex. 6. From $9ab - 7de + 8eg$ take $3eg - 7de - 9ab$.

Ex. 7. From the sum of $3ab + 4cd - 5acx$, and $2cd - 4bx + 3ab - 2cd$, take $7ab - 3bx + 8cd$.

Ex. 8. From $\frac{1}{3}a + \frac{1}{2}b$, take $\frac{1}{2}a - \frac{1}{2}b$.

Ex. 9. From $9ab - 7dx + 8ey$, take the sum of
$3ab + 7ey + 11dx$ and $9dx - 6ab - 3ey$.

Ex. 10. From the sum of $3bx + 4ay - 15bc + 20$, and $35 + 7ay + 4bx$, take the sum of $3 + 8ay - 7bx$, $3bc + 8ay + 2bx$ and $15bx - 32 - 8bc$.

SECTION III.

Multiplication.

16. It has already been said (4) that the product of two quantities, such as a and b, is expressed by $a \times b$, $a \cdot b$, or more simply by ab.

In the same manner the product of any number of factors is expressed. Thus, $a \times b \times c \times d$ is written $abcd$.

So likewise, $5\,ab \times 3\,cd = 5\,ab\,3\,cd$. But since it is indifferent what order is maintained amongst the factors, the result may be written

$$5 \times 3 \times abcd, \text{ or } 15\,abcd.$$

Hence, to multiply monomials, (or expressions consisting of but one term,) *we multiply the numerical parts,* or coefficients, *and to the product annex the product of the literal parts.*

EXAMPLES.

Multiply	$4\,ac$ by $3\,bd$.	*Ans.*	$12\,abcd$.
"	$3\,ad$ by $5\,ac$.	"	$15\,aacd$.
"	$4\,aax$ by $7\,ay$.	"	$28\,aaaxy$.
What is the value of	$7\,ay \times 12\,bx \times 6\,ab$	"	$504\,aabbxy$.

Reduce the following, viz.:

$6\,ax \times 3\,ay \times 4\,bc =$

$8\,by \times 6\,abx \times 2\,cd =$

$5\,aaabbbx \times 7\,aaabxx =$

$7\,abxxy \times 3\,abxx \times 4\,x =$

$12\,aabc \times 3\,abccc =$

$16\,aad \times 2\,bbc \times 7\,abcd =$

$13\,aabcc \times 6\,abcx =$

$5\,aac \times 4\,aabbc \times 7\,aabx =$

17. In the above examples we have frequently met with such expressions as aaa, bb, &c.

Now we have learned in arithmetic that the
product of two equal factors is the square of one of them,
" three " " cube, "
" four " " fourth power, &c.

Consequently aa is the square or 2d power of a.
aaa is the cube or 3d power of a, &c.

In order to render the expressions more concise, the number of factors is indicated by putting a small figure over the root, and a little to the right; thus, a^2 is written for *aa*, and is read *a's* square, or the square of *a*.

Similarly $a^3 = aaa$, and is read *a's* cube, or the cube of *a*, so a^4, a^6, a^7 are respectively the same as *aaaa*, *aaaaaa*, and *aaaaaaa*; and are read *a's* fourth, *a's* sixth, and *a's* seventh power.

18. The figure which thus indicates the power, is called the *exponent*, or *index*, and represents the number of equal factors that are multiplied together. Thus, when we say

$$4^3 = 64, \text{ we mean } 4 \times 4 \times 4 = 64.$$

The *indices* must be carefully distinguished from the *coefficients*, since these express only successive additions, while the former represent successive multiplications. Thus,

$$3\,a = a + a + a, \text{ while } a^3 = a \times a \times a.$$

19. From what has been said above, it is easy to write the results in the last article more concisely. The second, third, and fourth may be written thus, $15\,a^2cd$, $28\,a^3xy$ and $504\,a^2b^2xy$.

The student will thus simplify the remaining results in that article.

20. Since $x^4 = xxxx$, and $x^5 = xxxxx$

it is evident that $x^4 \times x^5 = xxxx \times xxxxx$

$$= xxxxxxxxx = x^9.$$

Similarly we should find that

$$x^2 \times x^3 = x^5,\ x^7 \times x^5 = x^{12}, \text{ \&c.}$$

Hence, to multiply different powers of the same root we add their indices.

EXAMPLES.

Ex. 1. $7\,a^2 \times 5\,a^3 = 35\,a^5$.

Ex. 2. $5\,a^3x^2 \times 4\,a^2x^3 = 20\,a^5x^5$.

Ex. 3. $3\,a^2x^4 \times 6\,a^3x = 18\,a^5x^5$.

Ex. 4. $9\,a^4x^2 \times 7\,a^3x =$

Ex. 5. $7\,ab^3c^2 \times 4\,a^3bc^3 =$

Ex. 6. $12\,a^4bc^4 \times 3\,a^5bc^5 =$

Ex. 7. $3\,a^2x^3b \times 5\,ax^4by^4 =$

Ex. 8. $15\,a^3bx^4 \times 3\,a^2x^9b^7 =$

Ex. 9. $12\,a^5b^4c^3 \times 4\,a^{10}b^2c^7 =$

Ex. 10. $17\,a^2b^3c^4 \times ac^3 =$

Ex. 11. $8\,x^2y^3z \times 9\,xy^4z^3 =$

Ex. 12. $15\,a^3x^2y^5 \times 3\,b^7x^6y^8 =$

Ex. 13. $9\,b^2d^3y^2 \times 4\,a^3d^2 =$

Ex. 14. $15\,b^3d^2e^4 \times 5\,b^8d^9e^{10} =$

MULTIPLICATION OF POLYNOMIALS.

21. If $a + b$ is to be multiplied by any number as 3, it is equivalent to adding three quantities, each equal to $a + b$; the result will evidently be $3\,a + 3b$, which consists of three times the first quantity plus three times the second. Had the expression been $a - b$, the result would have been $3\,a - 3b$. This would be equally true if the multiplicand consisted of more than two terms, or if the multiplier were any other number. Hence,

To multiply a polynomial by a positive multiplier, we multiply each term separately, and connect the results by the signs with which the several terms were affected in the multiplicand.

NOTE.—To indicate that several quantities are to be affected by one sign we enclose them within brackets ( ), or place a vinculum, $\overline{\quad}$, over them. Thus, $(6 + 4) \times 5$ is equivalent to $10 \times 5 = 50$, while $6 + 4 \times 5 = 6 + 20 = 26$. So, also, $(a + b)^2$ or $\overline{a + b}^2$ is the square of $a + b$, while $a + b^2$ is equal to $b^2 + a$.

EXAMPLES.

Ex. 1. $(5ax - 6a^3b - 3ac) \times 4a = 20a^2x - 24a^4b - 12a^2c$.

Ex. 2. $(7xy^2 - 4az + 3b) \times 6a^2x^2 = 42a^2x^3y^2 - 24a^3x^2z + 18\,a^2bx^2$.

Ex. 3. $(4\,ab^2 - 5\,a^2c + b^2) \times 7\,a^2b^3c =$

Ex. 4. $(8\,a^2x - 15\,ax^2 + 15) \times 12\,a^3x^3 =$

Ex. 5. $(9\,bc^3 - 8\,b^3c + b^2c^2) \times 15\,b^2c =$

Ex. 6. $(12\,ab^3 - 4\,d^2x - 5\,a^2c) \times 4\,ab^2cx =$

Ex. 7. $(3\,a^2b^2 - \frac{1}{2}\,b^3c + 8\,c^2) \times \frac{1}{2}\,bc^4 =$

Ex. 8. $(9\,a^3d - 4\,ad^2 - 3\,d) \times 7\,a^4b =$

Ex. 9. $(\frac{3}{2}\,x^3y - \frac{4}{3}\,b^3z + \frac{3}{4}\,xy^3) \times 12\,b^3x^3y^3 =$

In the above cases we perceive that a negative quantity multiplied by a positive, gives a negative product.

22. We shall now proceed to the case in which the multiplier is a polynomial as well as the multiplicand.

Let it be required to multiply $x + y$ by $a + b$. This is evidently requiring us to increase $x + y$, $a + b$ times, which is equivalent to multiplying it by a and also by b and adding the results. The operation may be arranged thus:

$$\begin{array}{rl}
x + y & \\
a + b & \\
\hline
ax + ay & \text{product by } a \\
+ bx + by & \quad \text{"} \quad b \\
\hline
ax + bx + ay + by & \quad \text{"} \quad a + b.
\end{array}$$

Similarly if the product of $(x - y)$ by $(a + b)$ were required, the operation would evidently be

$$\begin{array}{rl}
x - y & \\
a + b & \\
\hline
ax - ay & \text{product by } a \\
bx - by & \quad \text{"} \quad \text{by } b \\
\hline
ax - ay + bx - by & \quad \text{by } (a + b).
\end{array}$$

23. Had the multiplier been $a - b$ it is evident the first line $ax - ay$, which is a times the multiplicand, would have been too great; and would require to be diminished by b times $(x - y)$ or $bx - by$, which is the second line. But as in subtraction, we change the signs of the subtracting terms, the operation by addition might still be preserved, by writing the terms in the last-mentioned line, with the opposite signs, as below.

$$\begin{array}{rl}
x - y & \\
a - b & \\
\hline
ax - ay & \text{product by } a \\
- bx + by & \quad \text{"} \quad -b \\
\hline
ab - ay - bx + by & \quad \text{"} \quad (a - b).
\end{array}$$

24. By examining the various terms in this operation we perceive that

$$\begin{array}{l}
a \times x = + ax \\
a \times - y = - ay \\
- b \times x = - bx \\
- b \times - y = + by
\end{array}$$

Hence we derive the following

Rule for Signs.

Like signs in multiplication produce *plus*; *unlike signs* in multiplication produce *minus*.

EXAMPLES OF THE MULTIPLICATION OF POLYNOMIALS.

Ex. 1.
$$\begin{array}{lll} a+b & & \\ a+b & & \\ \hline a^2+ab & & \\ & ab & +b^2 \\ \hline a^2+2ab & & +b^2 \\ \hline \end{array}$$

Ex. 2.
$$\begin{array}{lll} a-b & & \\ a-b & & \\ \hline a^2-ab & & \\ & -ab & +b^2 \\ \hline a^2-2ab & & +b^2 \\ \hline \end{array}$$

Ex. 3.
$$\begin{array}{lll} a+b & & \\ a-b & & \\ \hline a^2+ab & & \\ & -ab & -b^2 \\ \hline a^2 & & -b^2 \\ \hline \end{array}$$

Ex. 4.
$$\begin{array}{llll} a^2-2ab+b^2 & & & \\ a+b & & & \\ \hline a^3 & -2a^2b & +ab^2 & \\ & a^2b & -2ab^2 & +b^3 \\ \hline a^3 & -a^2b & -ab^2 & +b^3 \\ \hline \end{array}$$

Ex. 5.
$$\begin{array}{lllll} 3a^3b-2ab^3+b^4 & & & & \\ 2ab+b^2 & & & & \\ \hline 6a^4b^2 & & -4a^2b^4 & +2ab^5 & \\ & +3a^3b^3 & & -2ab^5 & +b^6 \\ \hline 6a^4b^2 & +3a^3b^3 & -4a^2b^4 & & +b^6 \\ \hline \end{array}$$

Ex. 6. Multiply x^2+y^2 by x^2-y^2. *Ans.* x^4-y^4.

Ex. 7. Multiply $x^3-3x^2y+3xy^2-y^3$ by $x^2-2xy+y^2$.

Ans. $x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5$.

Ex. 8. Multiply $3a^4-7a^3b+b^3$ by $2a^2-4ab$.

Ans. $6a^6-26a^5b+28a^4b^2+2a^2b^3-4ab^4$.

Ex. 9. Multiply $x^2+2xy+y^2$ by $x^2-2xy+y^2$.

Ans. $x^4-2x^2y^2+y^4$.

Ex. 10. Multiply $x^4 + x^3y + x^2y^2 + xy^3 + y^4$ by $x - y$.
Ans. $x^5 - y^5$.

Ex. 11. Square $3x^2 - 4ax$.
Ans. $9x^4 - 24ax^3 + 16a^2x^2$.

Ex. 12. Square $a^2 - 2ax + x^2$.
Ans. $a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4$.

Ex. 13. Cube $x + y$.
Ans. $x^3 + 3x^2y + 3xy^2 + y^3$.

Ex. 14. Determine the fifth power $a^3 + b^2$.
Ans. $a^{15} + 5a^{12}b^2 + 10a^9b^4 + 10a^6b^6 + 5a^3b^8 + b^{10}$.

Ex. 15. Multiply the square of $a + b$ by the cube of $a - b$.
Ans. $a^5 - a^4b - 2a^3b^2 + 2a^2b^3 + ab^4 - b^5$.

Ex. 16. Determine the product of the four factors $a - 4x$, $a - x$, $a + x$ and $a + 4x$
Ans. $a^4 - 17a^2x^2 + 16x^4$.

Ex. 17. Multiply $3a^4 - 7a^3b + 8a^2b^2 - 5b^4$ by $2a^2 - 3ab + b^2$.
Ans. $6a^6 - 23a^5b + 40a^4b^2 - 31a^3b^3 - 2a^2b^4 + 15ab^5 - 5b^6$.

25. It is often found that the different terms in the multiplicand, and also in the multiplier, contain the same number of factors. Thus in the last example of the preceding article, the literal portions of the several terms of the multiplier are $a^4 = aaaa$, $a^3b = aaab$, $a^2b^2 = aabb$ and $b^4 = bbbb$, each being composed of four factors. So in the multiplier, the literal part of each term contains two factors.

Quantities of this kind are said to be *homogeneous*.

In all cases where there are not more than two letters employed, and the several terms of the multiplicand and also of the multiplier are homogeneous, the operation may be shortened by omitting the letters until the close.

In arranging the terms, we must be careful to place them according to the powers of some letter, beginning either at the highest or the lowest, and regularly descending or ascending. Should any term in the regular series of powers be wanting, its coefficient must be supplied by a zero. In example 17, above referred to, the series of powers of a is

$$a^4, a^3, a^2;$$

a^1 being omitted, we must therefore conceive the term $0\,ab^3$ to be placed between $8\,a^2b^2$ and $-5\,b^4$. The complete series of coefficients is therefore

$$\begin{array}{ll}
\text{in the multiplicand} & 3-7+8+0-5 \\
\text{in the multiplier} & 2-3+1 \\
\hline
\text{multiplying these} & 6-14+16+0-10 \\
\text{we obtain} & \quad -9+21-24+0+15 \\
 & \qquad\qquad 3-7+8+0-5 \\
\hline
 & 6-23+40-31-2+15-5
\end{array}$$

and supplying the letters we have for the product

$6\,a^6-23\,a^5b+40\,a^4b^2-31\,a^3b^3-2\,a^2b^4+15\,ab^5-5\,b^6$

as before.

This is called *multiplication by detached coefficients.*

EXAMPLES.

Ex. 1. Multiply $x^3-3\,x^2y+y^3$ by x^2-y^2.

Supplying zeroes for the coefficients of the missing terms we have

$$\begin{array}{l}
1-3+0+1 \\
1+0-1 \\
\hline
1-3+0+1 \\
\quad -1+3-0-1 \\
\hline
1-3-1+4-0-1
\end{array}$$

and the product is $x^5-3\,x^4y-x^3y^2+4\,x^2y^3-y^5$.

Ex. 2. Cube $a+3\,b$ by this method.

The operation is

$$\begin{array}{l}
1+3 \\
1+3 \\
\hline
1+3 \\
\quad +3+9 \\
\hline
1+6+9 \\
1+3 \\
\hline
1+6+9 \\
\quad +3+18+27 \\
\hline
1+9+27+27
\end{array}$$

Hence the result is $a^3+9\,a^2b+27\,ab^2+27\,b^3$.

Ex. 3. Multiply $5\,x^3-3\,ax^2+5\,a^2x-a^3$ by $a^2+3\,ax+5\,x^2$.

The coefficients in the multiplier must be reversed, the operation will therefore be

$$\begin{array}{rrrrrr}
5 - & 3 + & 5 - & 1 & & \\
5 + & 3 + & 1 & & & \\
\hline
25 - & 15 + & 25 - & 5 & & \\
 & 15 - & 9 + & 15 - & 3 & \\
 & & 5 - & 3 + & 5 - & 1 \\
\hline
25 + & 0 + & 21 + & 7 + & 2 - & 1
\end{array}$$

And the product is $25\,x^5 + 21\,a^2x^3 + 7\,a^3x^2 + 2\,a^4x - a^5$.

Ex. 4. Multiply $3\,a^2 - 5\,ax + 2\,x^2$ by $2\,a^2 - 6\,ax - 3\,x^2$.
Ans. $6\,a^4 - 28\,a^3x + 25\,a^2x^2 + 3\,ax^3 - 6\,x^4$.

Ex. 5. Multiply $1 + 2\,x + 3\,x^2 + 4\,x^3 + 5\,x^4$ by $1 - x$.
Ans. $1 + x + x^2 + x^3 + x^4 - 5\,x^5$.

Ex. 6. Multiply $x^3 - 3\,x^2 + 3\,x - 1$ by $x^2 - 2\,x + 1$.
Ans. $x^5 - 5\,x^4 + 10\,x^3 - 10\,x^2 + 5\,x - 1$.

Ex. 7. Multiply $a^3 + 3\,a^2b + 3\,ab^2 + b^3$ by $a^3 - 3\,a^2b + 3ab^2 - b^3$.
Ans. $a^6 - 3\,a^4b^2 + 3\,a^2b^4 - b^6$.

Ex. 8. Raise $a - b$ to the fifth power by this process.
Ans. $a^5 - 5\,a^4b + 10\,a^3b^2 - 10\,a^2b^3 + 5\,ab^4 - b^5$.

Ex. 9. Square $a^3 - 3\,a^2y + y^3$.
Ans. $a^6 - 6\,a^5y + 9\,a^4y^2 + 2\,a^3y^3 - 6\,a^2y^4 + y^6$.

SECTION IV.

Division.

26. The division of simple quantities can present but little difficulty, since its operations must be the reverse of those of multiplication.

Thus the product of two powers of the same root is found by adding their indices. For example, $a^7 \times a^3 = a^{10}$. Hence, $a^{10} \div a^7 = a^3$, and as the operation will be similar whatever the indices may be, it follows that

To divide different powers of the same root, subtract the index of the divisor from that of the dividend, the remainder is the index of the quotient.

We shall also find that the same rule for signs holds as in multiplication.

Hence, in *Multiplication* and *Division*

Like signs produce *plus*,

Unlike signs produce *minus*.

NOTE.—The general value of $\frac{x^m}{x^n}$ is x^{m-n}. If in this we suppose $n = m$ we shall have

$$\frac{x^m}{x^m} = x^{m-m}$$

$$\text{or } 1 = x^0,$$

whence the 0th power of any number is equal to 1.

Again, if in $\frac{x^m}{x^n} = x^{m-n}$ we make $m = 0$

we shall have $\frac{x^0}{x^n} = x^{0-n}$ or $\frac{1}{x^n} = x^{-n}$.

27. It is often convenient for beginners to write the divisor beneath the dividend as in a fraction, and cancel the like factors, as in arithmetic.

Thus the division of

$$27\,a^4b^3c^2 \text{ by } -9\,a^2b^3c$$

may be performed thus,

$$\frac{27\,a^4b^3c^2}{-9\,a^2b^3c} = -3\,a^2c$$

the common factors 9, a^2, b^3, and c having been cancelled.

This mode of operation can hardly be recommended, however, except for those persons who have not acquired any facility in calculation, as we may obtain the result in all cases, at least where the quotient is not fractional, by a more simple process. We should divide the coefficients, and then the literal parts, setting them down in order: first, however, having been careful to notice and write the sign with which the quotient will be affected.

In the above example the operation would be as follows:

$$-9\,a^2b^3c\,)\,\underline{27\,a^4b^3c^2}$$
$$-3\,a^2c.$$

Thus unlike signs produce minus; 9 into 27 gives 3, a^2 into a^4 goes a^2, b^3 into b^3 gives 1, and c into c^2, goes c times. The result is therefore as above; the factor 1 not appearing, as it does not affect the result.

EXAMPLES.

Ex. 1. Divide $-35\,a^2b^3$ by $-5\,ab^2$. *Ans.* $7\,ab$.
Ex. 2. Divide $-15\,a^3bx^2$ by $3\,a^2bx$. *Ans.* $-5\,ax$.
Ex. 3. Divide $21\,a^5b^3c$ by $7\,a^3b^2$.
Ex. 4. Divide $18\,b^3x^2$ by $-6\,bx$.
Ex. 5. Divide $-36\,a^3b^2cx^2$ by $3\,ab^2cx$.
Ex. 6. Divide $-9\,ac^2yx$ by $3\,ac^2x$.
Ex. 7. Divide $-d^2cx^3$ by $-dcx$.
Ex. 8. Divide $-11\,b^3c^4x^7$ by b^3cx^2.
Ex. 9. Divide $-13\,b^2c^3y^8$ by $-b^2cy^7$.

28. It frequently happens that the divisor is not contained exactly in the dividend. In such cases the quotient can only be expressed by a fraction; and the method first pointed out above is the most concise.

Thus, let it be required to divide

$$-15\,a^4x^3y \text{ by } 10\,a^2x^4y^2.$$

The quotient would be represented by the fraction

$$-\frac{15\,a^4x^3y}{10\,a^2x^4y^2}$$

which, by cancelling the common factors 5, a^2, x^3, and y, is reduced to

$$-\frac{3\,a^2}{2\,xy}.$$

Again, $21\,x^2y^3z \div -14\,a^3xy^3 = -\dfrac{21\,x^2y^3z}{14\,a^3xy^3} = -\dfrac{3\,xz}{2\,a^3}$,
the factors 7, x, and y^3 having been stricken out.

EXAMPLES.

Ex. 1. Divide $15\,a^3x^2$ by $-3\,a^2x^3$. *Ans.* $-\dfrac{5\,a}{x}$.
Ex. 2. Divide $-17\,a^2bx^3$ by $-3\,a^3bx$. *Ans.* $\dfrac{17\,x}{3\,a}$.
Ex. 3. Divide $-21\,b^8cx$ by $17\,bcx^3$.
Ex. 4. Divide $-33\,a^3by^2$ by $-22\,b^2y^8$.
Ex. 5. Divide $-29\,b^3c^2y^2$ by $-14\,ab^3c^2$.
Ex. 6. Divide $35\,a^3bx^2$ by $15\,ab^2x^3$.
Ex. 7. Divide $27\,a^4bc^2$ by $-6\,a^2b^3c^6$.

Ex. 8. Divide $-19\,a^2b^3x^4$ by b^3c^2.

Ex. 9. Divide x^2yz^3 by $5\,xy^4z$.

Ex. 10. Divide $-8\,x^3bc^2$ by $-18\,b^3cx^4$.

29. When the dividend consists of several terms, these must be divided separately, and the several quotients connected by their proper signs.

Thus, $(72\,a^2x^3 - 16\,a^5x^4 + 64\,a^3x^5) \div 8\,a^2x^2$

is equal to $9\,x - 2\,a^3x^2 + 8\,ax^3$.

The reason of this rule is too evident to need explanation.

Examples.

Ex. 1. Divide $16\,a^4x^3 - 24\,a^3x^4$ by $8\,a^2x^2$.

Ans. $2\,a^2x - 3\,ax^2$.

Ex. 2. Divide $27\,a^3bc - 16\,ab^2c^2$ by $9\,a^2b$.

Ans. $3\,ac - \dfrac{16\,bc^2}{9\,a}$.

Ex. 3. Divide $9\,a^3bx - 4\,ab^2x^3 + 12\,a^2bx^2$ by $3\,a^2bx$.

Ex. 4. Divide $15\,x^3y^2 - 27\,xy^3 + 14\,x^2y^4$ by $7\,xy^3$.

Ex. 5. Divide $4\,a^3b + 5\,b^3$ by $2\,a^2b^2$.

Ex. 6. Divide $15\,ab^3x - 14\,abx^3 + 25\,a^3bx$ by $5\,a^2b$.

30. The division of polynomials is performed in the same manner as long division in arithmetic, applying the principles laid down in the preceding pages.

In all cases the several terms of the divisor and the dividend must be arranged according to the powers of some one letter, either beginning with the highest and regularly descending, or with the lowest and regularly ascending.

Having so arranged them, *divide the first term of the divisor into the first term of the dividend, for the first term of the quotient. Multiply the divisor by the term thus determined, and subtract the product from the dividend, arranging the terms as above directed.*

Divide the first term of the remainder by the first term of the divisor, and so proceed until the operation is accomplished.

EXAMPLES.

Ex. 1.
$$\begin{array}{l} a - x)\,a^2 - 2ax + x^2\,(a - x \\ \underline{a^2 - \quad ax} \\ \qquad - ax + x^2 \\ \qquad \underline{- ax + x^2} \end{array}$$

Ex. 2.
$$\begin{array}{l} a + x)\,a^2 - x^2\,(a - x \\ \underline{a^2 + ax} \\ \quad - ax - x^2 \\ \quad \underline{- ax - x^2} \end{array}$$

Ex. 3.
$$\begin{array}{l} x + y)\,x^3 - y^3\,\left(x^2 - xy + y^2 - \dfrac{2y^3}{x + y}\right. \\ \underline{x^3 + x^2y} \\ \quad - x^2y - y^3 \\ \quad \underline{- x^2y - xy^2} \\ \qquad\qquad xy^2 - y^3 \\ \qquad\qquad \underline{xy^2 + y^3} \\ \qquad\qquad\quad - 2y^3 \end{array}$$

Ex. 4.
$$\begin{array}{l} 3a^3b - 2ab^3 + b^4)\,6a^4b^2 + 3a^3b^3 - 4a^2b^4 + b^6\,(2ab + b^2 \\ \qquad\qquad \underline{6a^4b^2 - 4a^2b^4 + 2ab^5} \\ \qquad\qquad\qquad 3a^3b^3 - 2ab^5 + b^6 \\ \qquad\qquad\qquad \underline{3a^3b^3 - 2ab^5 + b^6} \end{array}$$

Ex. 5.
$$\begin{array}{l} x - y)\,x^5 - y^5\,(x^4 + x^3y + x^2y^2 + xy^3 + y^4 \\ \underline{x^5 - x^4y} \\ \quad x^4y - y^5 \\ \quad \underline{x^4y - x^3y^2} \\ \qquad x^3y^2 - y^5 \\ \qquad \underline{x^3y^2 - x^2y^3} \\ \qquad\quad x^2y^3 - y^5 \\ \qquad\quad \underline{x^2y^3 - xy^4} \\ \qquad\qquad xy^4 - y^5 \\ \qquad\qquad \underline{xy^4 - y^5} \end{array}$$

Ex. 6. Divide $a^3 + b^3$ by $a + b$.

Ans. $a^2 - ab + b^2$.

Ex. 7. Divide $a^4 - 2a^2x^2 + x^4$ by $a^2 + 2ax + x^2$.

Ans. $a^2 - 2ax + x^2$.

Ex. 8. Divide $a^6 - 3a^4x^2 + 3a^2x^4 - x^6$ by $a^3 + 3a^2x + 3ax^2 + x^3$.

Ex. 9. Divide $24a^4 - b^4$ by $3a - 5b$.

Ex. 10. Divide $6x^6 - 6y^6$ by $2x^2 - 2y^2$.

Ex. 11. Divide $x^7 - y^7$ by $x - y$.

Ex. 12. Divide $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$ by $a^2 - 2ab + b^2$.

Ex. 13. Divide $12 - 4y - 3y^2 + y^3$ by $4 - y^2$.

Ex. 14. Divide $81x^4 - 18x^2 + 1$ by $9x^2 - 6x + 1$.

Ex. 15. Divide $48x^3 - 76ax^2 - 64a^2x + 105a^3$ by $2x - 3a$.

Ex. 16. Divide $x^6 + y^6$ by $x + y$.

Ex. 17. Divide 1 by $1 + x$.

Ex. 18. Divide $1 + x$ by $1 - x$.

Ex. 19. Divide $1 + x$ by $1 - 2x + x^2$.

Ex. 20. Divide 1 by $1 + 2x + x^2$.

31. If the terms of the divisor and also of the dividend be homogeneous, and do not contain more than two letters, the operation may be performed by detached coefficients. Thus:

Ex. 1. Divide $6a^4b^2 + 3a^3b^3 - 4a^2b^4 + b^6$ by $3a^3b - 2ab^3 + b^4$.

The coefficients are, supplying that of a^2 in the divisor, and of a in the dividend,

$$\begin{array}{r} 3+0-2+1)\ 6+3-4+0+1\ (2+1 \\ \underline{6+0-4+2} \\ 3+0-2+1 \\ \underline{3+0-2+1} \end{array}$$

Now the literal portions of the first terms being a^4b^2 and a^3b, that of the first term of the quotient is ab. Hence the complete quotient is

$$2ab + b^2$$

as in Ex. 4 of last article.

Ex. 2. Again let it be required to divide

$$6x^6 - 6y^6 \text{ by } 3x^4 + 3x^2y^2 + 3y^4$$

Here the coefficients are

$$\begin{array}{l} 3+0+3+0+3)\ 6+0+0+0+0+0-6\ (2+0-2 \\ \qquad\qquad\qquad\quad 6+0+6+0+6 \\ \qquad\qquad\qquad\qquad\quad 0-6+0-6+0-6 \\ \qquad\qquad\qquad\qquad\qquad -6+0-6+0-6 \end{array}$$

and $x^6 \div x^4 = x^2$. Hence the quotient is $2x^2 + 0\,xy - 2y^2 = 2x^2 - 2y^2$.

As examples of this method, the pupil can employ those of the last article, and thus more readily compare the two methods of proceeding.

SYNTHETIC DIVISION.

32. In the method of dividing by detached coefficients, the several coefficients of the divisor are successively multiplied by the various terms of the quotient, and the products subtracted from the partial dividends. Now, since in subtraction we change the signs of the subtrahend and then add; if we write the terms of the several products with their signs changed, each operation will become one of addition.

This may be done with facility by changing all the signs in the divisor, except the first, which must not be changed, on account of the liability to error in the sign of the quotient to which such change would lead. No difficulty can arise in the subtractions from the sign of the first term not being changed, for the first term of the product being always the same as that of the partial dividend, need not be written.

Thus let it be required to divide

$x^4 - 3\,ax^3 - 8\,a^2x^2 + 18\,a^3x - 8\,a^4$ by $x^2 + 2\,ax - 2\,a^2$

writing the coefficients, changing the signs of the second and third in the divisor, the operation becomes

$$\begin{array}{l} 1-2+2)\ 1-3-8+18-8\ (1-5+4 \\ \qquad\qquad\quad * -2+2 \\ \qquad\qquad\quad -5\ -6+18 \\ \qquad\qquad\qquad\ * +10-10 \\ \qquad\qquad\qquad\qquad 4+8-8 \\ \qquad\qquad\qquad\qquad\ * -8+8 \end{array}$$

and the quotient is $x^2 - 5\,ax + 4\,a^2$.

In examining the above process, we will readily see that the -6 and 18 in the third line, and $+8$ and -8 in the fourth might have been omitted; the operation would then stand thus:

$$\begin{array}{l} 1-2+2)1-3-8+18-8(1-5+4 \\ \qquad\qquad\;\; \underline{-2+2} \\ \qquad\qquad\;\; \underline{-5} \\ \qquad\qquad\qquad +\underline{10}-10 \\ \qquad\qquad\qquad\quad\; 4 \\ \qquad\qquad\qquad\quad\; \underline{-8+8} \\ \qquad\qquad\qquad\quad\;\; 0 \quad 0 \end{array}$$

Which may be more concisely written by placing the terms of the divisor in a vertical column to the left, thus:

	1	1	−3	−8	+18	−8
	−2		−2	+10	−8	
	2			2	−10	+8
first terms of dividend			−5	+4	0	0
quotient		1	−5	+4		

in which the several terms of each product are written in a diagonal line, downwards and to the right.

Again. Divide $2a^7 - 6a^4 + 4a^3 - 7a^2 + 9$ by $2a^3 + 6a^2 - 10$

Coeffits. of divisor.	2	2	0	0	−6	4	−7	0	+9
	−6		−6	+18	−54	+150	−372		
	0			0	0	0	0	0	
	10				10	−30	+90	−250	+620
1st terms div.			−6	+18	−50	+124	−289	−250	+629
Quotient		1	−3	+9	−25	+62			

Consequently the quotient is

$$a^4 - 3a^3 + 9a^2 - 25a + 62,$$

and the remainder is

$$-289a^2 - 250a + 629.$$

For further examples the pupil may solve those of art. 30, by this method.

TABLE OF USEFUL FORMULÆ.

$$a^2-x^2=(a+x)(a-x),$$
$$a^3-x^3=(a^2+ax+x^2)(a-x),$$
$$a^3+x^3=(a^2-ax+x^2)(a+x),$$
$$a^4-x^4=(a^2+x^2)(a^2-x^2),$$
$$=(a^2+x^2)(a+x)(a-x),$$
$$a^6-x^6=(a^3+x^3)(a^3-x^3)=(a^2+x^2)(a^2+ax+x^2)(a-x)$$
$$=(a^3-x^3)(a^2-ax+x^2)(a+x)$$
$$=(a+x)(a-x)(a^2+ax+x^2)(a^2-ax+x^2)$$
$$=(a^2-x^2)(a^4+a^2x^2+x^4)$$

$$a^4+a^2x^2+x^4=(a^2-ax+x^2)(a^2+ax+x^2)$$

$$\frac{a^2-b^2}{a-b}=a+b, \qquad \frac{a^2-b^2}{a+b}=a-b.$$

$$\frac{a^3-b^3}{a-b}=a^2+ab+b^2, \qquad \frac{a^3+b^3}{a+b}=a^2-ab+b^2.$$

$$\frac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3, \quad \frac{a^4-b^4}{a+b}=a^3-a^2b+ab^2-b^3.$$

$$a^0=1, \qquad a^{-m}=\frac{1}{a^m}, \qquad a^m=\frac{1}{a^{-m}}, \qquad a^mb^{-n}=\frac{a^m}{b^n}.$$

SECTION V.

Fractions.

33. The principles upon which the operations with algebraic fractions are performed, being the same as those already employed in arithmetic, we might at once proceed to apply them. As these principles, however, are more readily explained by the algebraic process, we shall enter at some length upon a few of them: and thus perhaps remove some difficulties which the pupil may have felt in his course heretofore.

34. One of the first reductions frequently required in working with fractions is to reduce them to their lowest terms. For this purpose we first find the common measure.

The rule for obtaining the *common measure* is founded on the following principles.

The *common measure* of two quantities, a and b, will be a *factor* of the remainder that is left, after dividing the less (as a) into the greater.

That this is true may be thus shown:

Let a be contained in b, m times, leaving the remainder r. Then we will evidently have

$$b = am + r.$$

Now, since the greatest common measure g, will measure b and a, if we divide the above equation by g, it will become

$$\frac{b}{g} = \frac{am}{g} + \frac{r}{g},$$

in which $\frac{b}{g}$ and $\frac{am}{g}$ are integers. If, then, r be not divisible by g, we shall have an integer equal to a fraction, which is impossible. As, then, $\frac{r}{g}$ is an integer, g must be a factor of r.

Let, now, r be divided into a, leaving a remainder r'. It is evident that g is a divisor of r'. If we thus continue the division until we find a remainder t, which will divide the preceding one without leaving any remainder, then will $t = g$.

The formula of the operation is as follows:

$$
\begin{array}{llllll}
a)\,b\,(m & & & & & \\
\;\; ma & & & & & \\
\;\; \overline{r}) & a\,(n & & & & \\
 & nr & & & & \\
 & \overline{r'}) & r\,(p & & & \\
 & & pr' & & & \\
 & & \overline{r''}) & r'\,(q & & \\
 & & & qr'' & & \\
 & & & \overline{t}) & r''\,(s & \\
 & & & & st & \\
 & & & & \overline{o} &
\end{array}
$$

the last remainder being t.

To show that $t = g$, we have the following equations:

$$b = ma + r.$$
$$a = nr + r'.$$
$$r = pr' + r''.$$
$$r' = qr'' + t.$$
$$r'' = st.$$

From these equations we perceive that r'', r', r, a and b, must each be measured by t. Now, we have shown that the greatest common measure of a and b, viz., g, must measure each of the remainders: it must measure t, and of course cannot be greater than t. And as t is a common measure of a and b, g cannot be less than t. Therefore $g = t$.

Therefore, to find the greatest common measure of two numbers, *divide the less into the greater.* If there be a remainder, *divide it into the last divisor, and so proceed until a remainder is found which will be contained exactly in the preceding divisor. Then this remainder is the greatest common measure.*

Thus let the greatest common measure of 246 and 272 be required.

The operation is as follows:

```
246) 272 (1
     246
     ---
      26) 246 (9
          234
          ---
           12) 26 (2
               24
               --
                2) 12 (6
                   12
                   --
```

2, being the last divisor, is the greatest common measure We may assure ourselves of this, by resolving the two numbers into their factors. Thus:

$$246 = 2 \times 3 \times 41$$

and

$$272 = 2 \times 2 \times 4 \times 17$$

41 and 131 being primes. We thus see that 2 is the common factor. The operation may be shortened by first cancelling any prime factor that is contained in either number, and not in both. Thus, if the common measure of 1015 and 2871 were required, we perceive at once that 5 is a factor of the first; and the sum of the digits in 2871 being a multiple of 9, 2871 is divisible by 9, and neither of the factors of 9 being contained in 1015, we may strike out this factor. By this means the numbers are reduced to

203 and 319,

the greatest common measure of which is 29.

35. If we could determine readily all the factors which

each of the given numbers contains, we could at once strike out those which did not belong to the common measure. To resolve a number into its factors, however, being frequently more laborious than to perform the operation for finding the common measure, in the preceding rule, this method is always to be preferred, unless the factors are at once discoverable from the forms of the numbers. When we are operating upon algebraical quantities however, the monomial factors may at once be determined by inspection, and should in all cases be stricken out.

Thus, if the common measure of $5a^3b^2c$ and $7a^2bx$ were required, we at once perceive that 5, 7, c and x can form no part of that common measure. The question is thus reduced to finding the common measure of a^3b^2 and a^2b, which is at once seen to be a^2b.

Again, let it be required to find the common measure of the two polynomials

$$3a^5 + 6a^4b + 3a^3b^2$$

and $$6a^3b + 12a^2b^2 + 12ab^3 + 6b^4.$$

Here we perceive that $3a^3$ is a factor of the first, and $6b$ of the second expression, and as these have a common factor, 3, this must form a part of the required common measure.

Striking out the factors $3a^3$ and $6b$, and proceeding with the reduced expressions as directed in the rule Art. 34, the operation will be as follows:

$$a^2 + 2ab + b^2)\ a^3 + 2a^2b + 2ab^2 + b^3\ (a$$
$$\underline{a^3 + 2a^2b + ab^2}$$
$$ab^2 + b^3$$

dividing by the factor b^2 this becomes

$$a + b)\ a^2 + 2ab + b^2\ (a+b$$
$$\underline{a^2 + ab}$$
$$+ ab + b^2$$
$$\underline{ab + b^2}.$$

Hence $3 \times (a + b) = 3a + 3b$ is the required common measure.

Examples.

Ex. 1. Find the greatest common measure of

$$12a^2 - 8a - 4$$

and $$20a^3b - 10a^2b - 15ab + 5b.$$

Omitting the factors 4 and $5b$, these expressions are reduced to

$$3a^2 - 2a - 1$$

and $$4a^3 - 2a^2 - 3a + 1.$$

But here we at once meet with a difficulty. Since $3a^2$ will not divide into $4a^3$, unless fractions be admitted, which in the present case is manifestly improper, the question might appear impossible. This difficulty is removed, however, by considering that inasmuch as the common measure can contain no simple factor, the introduction of such a factor into either divisor or dividend cannot affect the final result. In the above case, then, we may multiply the dividend by 3, and the operation will stand thus:

$$\begin{array}{rl}
 & 4a^3 - 2a^2 - 3a + 1 \\
 & 3 \\
\hline
3a^2 - 2a - 1) & 12a^3 - 6a^2 - 9a + 3\ (4a \\
 & 12a^3 - 8a^2 - 4a \\
\hline
 & \quad 2a^2 - 5a + 3 \\
 & \quad 3 \\
\hline
 & \quad 6a^2 - 15a + 9\ (2 \\
 & \quad 6a^2 - 4a - 2 \\
\hline
 & \qquad -11a + 11
\end{array}$$

or cancelling the factor -11

$$\begin{array}{rl}
a - 1) & 3a^2 - 2a - 1\ (3a + 1 \\
 & 3a^2 - 3a \\
\hline
 & \qquad a - 1 \\
 & \qquad a - 1 \\
\hline
\end{array}$$

and $a - 1$ is the common measure required.

From the above investigation the following rule may be derived.

To find the common measure of two polynomials, first determine, and divide by the monomial factors that either of them contains.

Divide that one of the reduced expressions in which the leading quantity is involved to the highest power, by the other; and cancel from the remainder any monomial factor that it contains.

Divide the reduced remainder into the preceding divisor, and so proceed until no remainder occurs; the last divisor multiplied by the common divisor of the monomials *stricken out from the original expression, will be the common measure required.*

If in the course of the operation, a monomial remainder occurs, the original expressions have no polynomial divisor.

Should the coefficient of the 1st term of any of the divisors not be contained in that of the first term of the dividend, the latter must be multiplied by such a factor as will make the division possible in integers.

36. To reduce a fraction to its lowest terms, the terms of that fraction must be divided by their greatest common measure.

EXAMPLES.

Ex. 1. Reduce $\frac{x^3+y^3}{x^2-y^2}$ to its lowest terms.

Ans. $\frac{x^2-xy+y^2}{x-y}$.

Ex. 2. Reduce $\frac{a^6-a^2x^4}{3a^6+3a^5x-3a^4x^2-3a^3x^3}$ to its lowest terms.

Ans. $\frac{a^2+x^2}{3a^2+3ax}$.

Ex. 3. Reduce $\frac{x^2+2xy+y^2}{x^3-xy^2}$ to its lowest terms.

Ans. $\frac{x+y}{x^2-xy}$.

Ex. 4. Reduce $\frac{3a^3-3a^2y+ay^2-y^3}{4a^2-5ay+y^2}$ to its lowest terms.

Ans. $\frac{3a^2+y^2}{4a-y}$.

Ex. 5. Reduce $\frac{2ax^2-a^2x-a^3}{2x^2+3ax+a^2}$ to its lowest terms.

Ans. $\frac{ax-a^2}{x+a}$.

Ex. 6. Reduce $\frac{x^4-y^4}{x^3-y^3}$ to its lowest terms.

Ans. $\frac{x^3-x^2y+xy^2-y^3}{x^2-xy+y^2}$.

Ex. 7. Reduce $\frac{15x^3-x^2-x+3}{9x^4+2x^2+1}$ to its lowest terms.

Ans. $\frac{5x+3}{3x^2+2x+1}$.

Ex. 8. Reduce $\frac{2x^3-16x-6}{3x^3-24x-9}$ to its lowest terms.

Ans. $\frac{2}{3}$.

Ex. 9. Reduce $\frac{48x^3+36x^2y-15y^3}{24x^3-22x^2y+17xy^2-6y^3}$ to its lowest terms. *Ans.* $\frac{24x^2+30xy+15y^2}{12x^2-5xy+6y^2}$.

Ex. 10. Reduce $\frac{5a^3+10a^2b+5ab^2}{8a^3+8a^2b}$ to its lowest terms.

Ans. $\frac{5a+5b}{8a}$.

Ex. 11. Reduce $\frac{6ac+10bc+9ad+15bd}{6c^2+9cd-2c-3d}$ to its lowest terms. *Ans.* $\frac{3a+5b}{3c-1}$.

Ex. 12. Reduce $\frac{9x^5+2x^3+4x^2-x+1}{15x^4-2x^3+10x^2-x+2}$ to its lowest terms. *Ans.* $\frac{3x^3+x^2+1}{5x^2+x+2}$.

Ex. 13. Reduce $\frac{a^3b+2a^2b^2+2ab^3+b^4}{5a^5+10a^4b+5a^3b^2}$ to its lowest terms. *Ans.* $\frac{a^2b+ab^2+b^3}{5a^4+5a^3b}$.

37. *To reduce a mixed number to a simple fractional expression.*

The principles by which this is performed, being identical with those with which the pupil has already become familiar in arithmetic, require no remark here, except that *when the fraction is negative, the numerator is to be subtracted from the product of the integer and denominator, instead of being added, as is always done in arithmetic.*

Examples.

Ex. 1. Reduce $a-\frac{b^2-a^2}{a}$ to a fraction.

The operation is a

× by a

a^2

subtract b^2-a^2

$2a^2-b^2$ = the numerator.

Hence $\frac{2a^2-b^2}{a}$ is the fraction required.

Ex. 2. Reduce $3a^2+x+\frac{4a^2b-5x}{b}$ to a fraction.

Here $(3a^2+x)b+4a^2b-5x=3a^2b+bx+4a^2b-5x$
$=7a^2b+bx-5x=$ the numerator,

and the fraction is $\frac{7a^2b+bx-5x}{b}$.

Ex. 3. Reduce $4ax-\frac{3ab}{y}$ to a fraction.

Ans. $\frac{4axy-3ab}{y}$.

Ex. 4. Reduce $7x-3y+\frac{x^2+y^2}{x-y}$ to a fraction.

Ans. $\frac{8x^2-10xy+4y^2}{x-y}$.

Ex. 5. Reduce $a+x-\frac{a^2+x^2}{a-x}$ to a fraction.

Ans. $\frac{2x^2}{x-a}$.

Ex. 6. Reduce $a^2-ax+x^2-\frac{x^3}{a+x}$ to a fraction.

Ans. $\frac{a^3}{a+x}$.

Ex. 7. Reduce $x+y-\frac{x^2-y^2+7}{x-y}$ to a fraction.

Ans. $\frac{7}{y-x}$.

38. The mode of performing the remaining operations with fractions being the same as in arithmetic, we shall merely annex examples for exercise.

Ex. 1. Reduce $\frac{a^3-b^3}{a+b}$ to a mixed number.

Ans. $a^2-ab+b^2-\frac{2b^3}{a+b}$.

Ex. 2. Reduce $\frac{4a^2-2ab+b^2}{2a+3b}$ to a mixed number.

Ans. $2a-2b+\frac{5b^2}{2a+3b}$.

Ex. 3. Reduce $\frac{6a^4+5ax-x^2}{3a^2+2ax}$ to a mixed number.

Ans. $2+\frac{ax-x^2}{3a^2+2ax}$.

Ex. 4. Reduce $\frac{a^6+b^6}{a+b}$ to a mixed number.

Ans. $a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5+\frac{2b^6}{a+b}$.

Ex. 5. Reduce $\frac{27a^3-3b^2-4x+9a^2}{9a^2}$

Ans. $3a+1-\frac{3b^2+4x}{9a^2}$.

Ex. 6. Reduce $\frac{3a}{2b}$, $\frac{7a^2}{4b^2}$, and $\frac{2b}{3a}$ to fractions having a common denominator.

Ans. $\frac{18a^2b}{12ab^2}$, $\frac{21a^3}{12ab^2}$, and $\frac{8b^3}{12ab^2}$.

Ex. 7. Reduce $\frac{a+x}{a-x}$ and $\frac{a-x}{a+x}$ to a common denominator.

Ans. $\frac{a^2+2ax+x^2}{a^2-x^2}$ and $\frac{a^2-2ax+x^2}{a^2-x^2}$.

Ex. 8. Reduce $\frac{4a}{a+x}$, $\frac{3b}{x^2}$, and $\frac{2ax}{a-x}$ to a common denominator.

Ans. $\frac{4a^2x^2-4ax^3}{a^2x^2-x^4}$, $\frac{3a^2b-3bx^2}{a^2x^2-x^4}$, and $\frac{2a^2x^3+2ax^4}{a^2x^2-x^4}$.

Ex. 9. Add $\frac{x+y}{x-y}$ and $\frac{x-y}{x+y}$.

Ans. $\frac{2x^2+2y^2}{x^2-y^2}$.

Ex. 10. Add $3a+\frac{2a+6}{3}$ and $5a+\frac{3a-2}{5}$.

Ans. $9a+1+\frac{4a+9}{15}$.

Ex. 11. From $\frac{7a-6}{3}$ take $\frac{9a+3}{5}$.

Ans. $\frac{8a-39}{15}$.

Ex. 12. From $6a+\frac{3x-2a}{a}$ take $2a+\frac{4a-3x}{x}$.

Ans. $4a+\frac{3x^2+ax-4a^2}{ax}$.

Ex. 13. From $\frac{a^2+b^2}{a-b}$ take $\frac{a^2-b^2}{a+b}$.

Ans. $\frac{2a^2b+2ab^2}{a^2-b^2}$.

Ex. 14. Collect the fractions $\frac{3a}{a-x}-\frac{a+x}{a}+\frac{a-x}{a+x}$.

Ans. $\frac{3a^3+2ax^2-x^3}{a^3-ax^2}$.

Ex. 15. Collect the fractions $\frac{4a}{13b}+\frac{6b}{13a}-\frac{7a}{2b}$ into one sum.

Ans. $\frac{12b^2-83a^2}{26ab}$.

Ex. 16. Subtract $\frac{a-b}{2}$ from $\frac{a+b}{2}$.

Ans. b.

Ex. 17. Reduce to one fraction $\frac{a^2}{a^2-x^2}-\frac{a}{a+x}+\frac{1}{a-x}$.

Ans. $\frac{ax+a+x}{a^2-x^2}$.

Ex. 18. Reduce $\frac{4}{3(1-x)^2}+\frac{7}{1-x}+\frac{2}{3(1+x)}-\frac{1+2x}{1-x^2}$ to one fraction.

Ans. $\frac{24-8x-13x^2}{3+3x-3x^2-3x^3}$.

Ex. 19. Reduce $\frac{a}{x(a-x)}+\frac{x}{a(a+x)}-\frac{a+x}{a^2-x^2}$ to a single fraction.

Ans. $\frac{a^2+ax+x^2}{a^2x+ax^2}$.

Ex. 20. Reduce $\frac{2}{a^2-x^2}-\frac{a+x}{a^3-x^3}+\frac{1}{a^2+ax+x^2}$ to a single fraction.

Ans. $\frac{2a^2}{(a^2-x^2)(a^2+ax+x^2)}$.

In solving the last few questions, and also those which follow, the pupil will find advantage in consulting the table of factors at the end of Art. 32, page 39.

Ex. 21. Multiply $\frac{a^2-x^2}{a^3+x^3}$ by $\frac{a^2-ax+x^2}{a-x}$.

Ans. 1.

Ex. 22. Multiply $\frac{3a}{4bx}$, $\frac{6x^2}{5a^2}$, and $\frac{15ax}{3b^2}$.

Ans. $\frac{9x^2}{2b^2}$.

Ex. 23. Multiply $\frac{a^2+b^2}{a^2-b^2}$ by $\frac{a-b}{a+b}$.

Ans. $\frac{a^2+b^2}{a^2+2ab+b^2}$.

Ex. 24. Multiply $\frac{x^2-9x+20}{x^2-6x}$ by $\frac{x^2-13x+42}{x^2-5x}$.

Ans. $\frac{x^2-11x+28}{x^2}$.

Ex. 25. Multiply $\frac{x(a^2+x^2)}{a^3-x^3}$, $\frac{a(a-x)}{a^3+a^2x+ax^2+x^3}$, and $\frac{a+x}{ax-x^2}$.

Ans. $\frac{a}{a^3-x^3}$.

Ex. 26. Multiply $\frac{a}{a+x}+\frac{x^2-ax}{a^2-x^2}$ by $\frac{a^2-x^2}{x}$.

Ans. $\frac{a^2-2ax+x^2}{x}$.

Ex. 27. Multiply $\frac{a+x}{a-x}+\frac{a-x}{a+x}$ by $\frac{a+x}{a-x}-\frac{a-x}{a+x}$.

Ans. $\frac{8a^3x+8ax^3}{(a^2-x^2)^2}$.

Ex. 28. Multiply $\frac{x^2-y^2}{a+b}$, $\frac{a^2-b^2}{xy+y^2}$, and $y+\frac{xy}{x-y}$.

Ans. $2ax-ay-2bx+by$.

Ex. 29. Divide $\frac{4x+12}{7}$ by $\frac{3x+9}{14a}$.

Ans. $\frac{8a}{3}$.

Ex. 30. Divide $\frac{3x+5}{a+b}$ by $\frac{15x+25}{a^2-b^2}$.

Ans. $\frac{a-b}{5}$.

Ex. 31. Divide $\frac{a-b}{a+b}$ by $\frac{a+b}{a-b}$.

Ans. $\frac{(a-b)^2}{(a+b)^2}$.

Ex. 32. Divide $\frac{x-y}{x+y}$ by $\frac{x^2-y^2}{x^2+2xy+y^2}$.

Ans. 1.

Ex. 33. Divide $b+\frac{b^2+bx}{b-x}$ by $b-\frac{b^2-bx}{b+x}$.

Ans. $\frac{b^2+bx}{bx-x^2}$.

Ex. 34. Divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$.

Ans. $x+\frac{b^2}{x}$.

Ex. 35. Divide $\frac{a+x}{a-x}+\frac{a-x}{a+x}$ by $\frac{a+x}{a-x}-\frac{a-x}{a+x}$.

Ans. $\frac{a^2+x^2}{2ax}$.

Ex. 36. Divide $\frac{8ab}{9x^2}+2+\frac{9x^2}{8ab}$ by $\frac{4a}{3x}+\frac{3x}{2b}$.

Ans. $\frac{2b}{3x}+\frac{3x}{4a}$.

This example is best performed by the rule for dividing polynomials.

CHAPTER III.

PROPORTION AND PROGRESSION.

38. *Ratio* is the relation which two quantities bear to each other in magnitude. It is expressed by the quotient arising from dividing the *second* by the *first*. Thus, the ratio of 4 to 5 is represented by the fraction $\frac{5}{4}$. The ratio of 12 to 3 is $\frac{3}{12}$ or $\frac{1}{4}$; of a to b is $\frac{b}{a}$ &c.

To indicate that two quantities are compared in this manner, we write them with two dots between them. Thus,

$$4:5,\ 12:3,\ a:b, \text{ \&c.};$$

which are read 4 to 5, 12 to 3, a to b, &c.

39. The 1st term of a ratio is the *antecedent*, the 2d the *consequent*.

40. When four quantities are such that the ratio of the 1st to the 2d is equal to that of the 3d to the 4th; they are said to be *proportionals;* and the series of terms forms a *proportion*.

Thus the numbers 3, 6, 8, and 16 are proportionals, the ratio of the 1st to the 2d, and of the 3d to the 4th being each equal to 2.

To express the equality of two ratios, we write them down with four dots (: :) between them. Thus,

$$3 : 6 :: 8 : 16;$$

which is read, as 3 is to 6 so is 8 to 16.

Cor. In every proportion the quotients of the 2d by the 1st, and of the 4th by the 3d, must evidently be equal.

So that $$a : b :: c : d$$

and $$\frac{b}{a} = \frac{d}{c} \text{ or } \frac{a}{b} = \frac{c}{d}$$

may be considered as convertible expressions, both indicating the equality of the ratios, a to b, and c to d.

41. Any number of quantities so related that the ratios of the successive pairs are all equal, are proportionals. Thus, 2, 6, 3, 9, 4, 12, 8, and 24 form a series of proportionals, the ratio being 3. Such a series is written

$$2 : 6 :: 3 : 9 :: 4 : 12 :: 8 : 24.$$

42. A series of *continual proportionals* is one in which every term has the same ratio to the succeeding one. Thus,

2, 4, 8, 16, 32, &c.,

are continual proportionals, the common ratio being 2.

A series of continual proportionals is likewise said to be in *geometrical progression*.

43. If four like quantities are proportionals, the product of the extremes is equal to that of the means; and conversely, if the product of any two quantities be equal to that of two others, the four are proportionals; those of one product being taken as extremes, and of the other as means.

Thus, if $a : b :: c : d$, then will $ad = bc$.
Or, conversely, if $ad = bc$, then $a : b :: c : d$.
For, since $a : b :: c : d$, we will evidently have

$$\frac{a}{b} = \frac{c}{d},$$ whence, clearing of fractions,

$$ad = bc.$$

Again, let $ad = bc$; then dividing by bd, we have

$$\frac{a}{b} = \frac{c}{d}.$$

Or, $$a : b :: c : d.$$

44. If three magnitudes be in continual proportion, the product of the extremes is equal to the square of the mean.

If $$a : b :: b : c, \text{ then } ac = b^2.$$

For, $$\frac{a}{b} = \frac{b}{c},$$

Multiply by bc, and $ac = b^2$.

45. If four quantities be proportionals, and any equal multiples be taken of the antecedents, and also of the consequents, the results will be proportionals.

If $a : b :: c : d$, then will $ma : nb :: mc : nd$.

For since $\frac{a}{b} = \frac{c}{d}$, we will have, by multiplying by $\frac{m}{n}$,

$$\frac{ma}{nb} = \frac{mc}{nd},$$

Or, $$ma : nb :: mc : nd.$$

Cor. This proposition is evidently true, if m or n should be fractions instead of whole numbers; so that the proposition might be extended to include any *parts* of the antecedents, and of the consequents.

46. If four quantities be proportional, they are proportionals by *division*, that is, the difference between the first and second is to either term, as the difference between the third and fourth is to the corresponding term.

Let $a : b :: c : d$, then $a \backsim b : a$ or $b :: c \backsim d : c$ or d.*

* To express the difference between two quantities, when it is not known which is the greater, the sign $\backsim$ is employed.

For, since $\frac{a}{b}=\frac{c}{d}$ we have $\frac{a}{b}\backsim 1=\frac{c}{d}\backsim 1$.

Or, $$\frac{a\backsim b}{b}=\frac{c\backsim d}{d}.$$

that is, $$a\backsim b : b :: c\backsim d : d.$$

Dividing the last equation by the following, viz., $\frac{a}{b}=\frac{c}{d}$,

it becomes $$\frac{a\backsim b}{a}=\frac{c\backsim d}{c}.$$

Or, $$a\backsim b : a :: c\backsim d : d.$$

47. If four quantities be proportional, they are proportional by *composition;* that is, the sum of the first and second is to either term as the sum of the third and fourth is to the corresponding term.

The demonstration of this, being almost identical with the last, can be supplied by the student.

Cor. From this and the preceding art., we have

$a+b : a\backsim b :: c+d : c\backsim d$, for, since $\frac{a+b}{b}=\frac{c+d}{d}$

and $\frac{a\backsim b}{b}=\frac{c\backsim d}{d}$, we have by division $\frac{a+b}{a\backsim b}=\frac{c+d}{c\backsim d}$.

Whence $a+b : a\backsim b :: c+d : c\backsim d$.

48. If four quantities be proportionals, they are proportionals when taken *inversely*. That is, the second is to the first as the fourth is to the third.

Let $$a : b :: c : d,$$

then $$\frac{a}{b}=\frac{c}{d}.$$

Whence $$\frac{b}{a}=\frac{d}{c} \text{ or } b : a :: d : c.$$

49. If four *like* quantities be proportional, they are proportional when taken *alternately;* that is, the first is to the third as the second is to the fourth.

Let $a : b :: c : d$, then $a : c :: b : d$; a, b, c, and d being like quantities.

For, since $\frac{a}{b}=\frac{c}{d}$, we have, by multiplying by $\frac{b}{c}$,

$$\frac{a}{c}=\frac{b}{d}.$$

Or, $a : c :: b : d.$

The restriction to like quantities is important, for we can have no ratio between any others. If, for instance, a and c were not like quantities, the expression $\frac{a}{c}$ would be an absurdity.

50. If the antecedents in one proportion be the same as those in another, then will one of the antecedents be to the sum of its consequents as the other antecedent is to the sum of its consequents.

Let $a : b :: c : d,$

and $a : e :: c : f,$

Then will $a : b + e :: c : d + f.$

For, we have $\frac{b}{a}=\frac{d}{c}$ and $\frac{e}{a}=\frac{f}{c}.$

Consequently, $\frac{b+e}{a}=\frac{d+f}{c}$, whence we readily conclude that $a : b + e :: c : d + f.$

Cor. 1. If we have $a : b :: c : d,$

and $e : b :: f : d,$ we shall in like manner have $a + e : b :: c + f : d.$

Cor. 2. These results would evidently be true whatever should be the number of proportions.

51. If any number of like magnitudes be proportional; as one antecedent is to its consequent, so is the sum of the antecedents to the sum of the consequents.

Let $a : b :: c : d :: e : f :: g : h.$

then will $a : b :: a + c + e + g : b + d + f + h.$

For we have by alternation (Art. 49)

$$a : c :: b : d$$

$$a : e :: b : f$$

$$a : g :: b : h$$

and $a : a :: b : b$

$\therefore$ (*Cor.* 2. Art. 50.) $a : a + c + e + g :: b : b + d + f + h$,
and alternately, $a : b :: a + c + e + g : b + d + f + h$.

52. If $a : b :: c : d$,
and $b : f :: d : h$,
then will $a : f :: c : h$.

For we have $\frac{a}{b} = \frac{c}{d}$

and $\frac{b}{f} = \frac{d}{h}$

$\therefore$ $\frac{ab}{bf} = \frac{cd}{dh}$ or $\frac{a}{f} = \frac{c}{h}$

and $a : f :: c : h$.

Cor. 1. This reasoning might evidently be extended to any number of proportions.

Cor. 2. From the above demonstration we have

$$ab : bf :: cd : dh.$$

Hence, if the corresponding terms of two proportions be multiplied together, the products will be proportional; and the proposition may evidently be extended to any number of proportions.

53. If we have any number of continued proportionals $a, b, c, d, \ldots n$ to m terms, then will $a : n :: a^{m-1} : b^{m-1}$.

For we evidently have $a : b :: a : b$
$a : b :: b : c$
$a : b :: c : d$
&c. &c.
to $m - 1$ proportions

$\therefore$ (*Cor.* 2. Art. 52.) $a^{m-1} : b^{m-1} :: abc \ldots : bc \ldots n$
$:: a : n$

54. If four quantities be proportional, like powers and roots of them will likewise be proportional.

Thus, if $a : b :: c : d$, then $a^n : b^n :: c^n : d^n$;

For, since $\frac{b}{a} = \frac{d}{c}$, $\frac{b^n}{a^n} = \frac{d^n}{c^n}$.

Or, $a^n : b^n :: c^n : d^n$.

SECTION II.

Arithmetical Progression.

55. When the terms of a series of quantities continually increase or decrease by the addition or subtraction of a given number, such series is said to be in *Arithmetical Progression.*

Thus the numbers 1, 4, 7, 10 . . ., which increase by the successive addition of 3, form an *increasing* arithmetical progression; 50, 47, 44, 41 . . . . which decrease by the subtraction of 3, form a *decreasing* arithmetical progression.

56. The number by which the successive terms of the series increase or diminish is called the *common difference.*

57. If a be the first term of an arithmetical progression, and d the common difference. Then the series will evidently be

if increasing $a,\ a+d,\ a+2d,\ a+3d,\ a+4d, \ldots$
if decreasing $a,\ a-d,\ a-2d,\ a-3d,\ a-4d. \ldots,$

By the inspection of the above series, we find that the coefficient of d, in any term, is a number less by unity than the number of the term in the series. Thus the coefficient of d in the fifth term is 4, in the sixth 5, &c.

The nth term will therefore be

$$a \pm (n-1)\,d.$$

In general we omit the double sign. This will lead to no want of generality in the results, if we consider the common difference in a decreasing series, *negative.*

58. The sum of the extremes is equal to the sum of any two terms equally distant from them.

Let $a, a+d, a+2d \ldots a+(n-3)\,d, a+(n-2)\,d, a+(n-1)\,d$, be a series of n terms. The sum of the extremes is

$$2a+(n-1)\,d.$$

And this will evidently be the sum of any two terms equally distant from them; and, likewise, twice the middle term, if the number of terms is odd.

59. This being the case, the sum of the series must be equal to the sum of the extremes multiplied by half the number of terms.

So that if S represent the sum of the series, a being the first, and l the last term, we shall have

$$S = (a + l)\frac{n}{2}.$$

This proposition may be otherwise demonstrated. Thus,

put $S = a + (a + d) + (a + 2d) + \ldots\ldots l.$

Writing the same series in an inverse order, we have

$S = l + (l - d) + (l - 2d) \ldots\ldots\ldots a.$

Adding this to the former, we obtain

$$\begin{aligned} 2S &= (a + l) + (a + l) \ldots\ldots \text{ to } n \text{ terms} \\ &= n(a + l), \end{aligned}$$

and $\quad S = \frac{n}{2}(a + l).$

Or supplying the value of l,

$$S = \frac{n}{2}(2a + (n - 1)d) \qquad \text{(A)}$$

This equation and the following,

$$l = a + (n - 1)d \qquad \text{(B)}$$

contain the whole theory of arithmetical progression.

Thus, if the first term, the last term, and the number of terms, are given to find the common difference, we have, from (B),

$$d = \frac{l - a}{n - 1}.$$

EXAMPLES.

Ex. 1. The first term is 5, the common difference 10, and the number of terms 50. Required the sum of the series?

Here $\quad l = a + (n - 1)d = 5 + 490 = 495,$

and $\quad S = \frac{n}{2}(a + l) = 25(5 + 495) = 12500.$

Ex. 2. A car, descending an inclined plane, moves 5 feet the first second, 15 the second, 25 the third, and so on, increasing 10 feet every second. How far will it move in a minute?

Here $a = 5, d = 10, n = 60.$

$$\therefore \quad S = \frac{n}{2}(2a + (n-1)d) = 30(10 + 590)$$

$$= 18000 \text{ feet} = 3\tfrac{9}{22} \text{ miles.}$$

This would be the rate down a plane which descended five feet in sixteen. No allowance being made for friction.

Ex. 3. Insert 10 arithmetical means between the numbers 3 and 58.

As there are 10 means there are 12 terms. Hence, the formula

$$l = a + (n-1)d$$

becomes $$58 = 3 + 11d.$$

$$\therefore \quad d = 5.$$

And the means are

8, 13, 18, 23, 28, 33, 38, 43, 48, and 53.

Ex. 4. The sum of a series, the first term, and the common difference being given, to find the number of terms.

This may be solved by the equation

$$S = \frac{n}{2}(2a + (n-1)d)$$

$$= na + \frac{(n^2 - n)d}{2},$$

clearing of fractions and transposing,

$$n^2 d + (2a - d)n = 2S,$$

a quadratic equation, which we are not at present in a situation to solve. See *Quadratic Equations.*

Ex. 5. What is the sum of the odd numbers

1, 3, 5, ... to 150 terms? *Ans.* 22500.

Ex. 6. The first term is 300, the common difference —4, and the number of terms 30. What is the sum of the series? *Ans.* 7260.

Ex. 7. The first term is 3, the common difference 7, and the number of terms 15. What is the sum of terms? *Ans.* 780.

Ex. 8. One hundred stones are placed in a straight line, at the distance of 3 yards: the first being 5 yards from a basket. How far will a person walk who shall bring them one by one to the basket. *Ans.* 17 miles and 780 yards.

Ex. 8. Insert 5 arithmetical means between 12 and 30.
Ans. 15, 18, 21, 24, and 27.

Ex. 9. A man being anxious to purchase a horse offers 1 dollar for the first nail in his shoes, 4 for the second, and so on in arithmetical progression. Now there being 8 nails in each shoe, what will the horse cost him?
Ans. \$1520.

Ex. 10. The first term is $\frac{1}{3}$, the common difference is $\frac{1}{3}$, and the number of terms 30. What is the sum?
Ans. 155.

Ex. 11. A and B start on a journey. A travels uniformly 40 miles per day. B goes 17 miles the first day, 20 the second, 23 the third, and so on in arithmetical progression. How far will they be apart at the end of twenty days?
Ans. 110 miles.

Ex. 12. What is the 16th term of the series 15, $\frac{44}{3}$, $\frac{43}{3}$, &c., and the sum of the 16 terms?
Ans. 10, and the sum 200.

Ex. 13. What is the nth term of the series $\frac{1}{3}, \frac{5}{6}, \frac{4}{3}$, &c., and the sum of the n terms?

Ans. nth term $\frac{1}{2}n-\frac{1}{6}$, and the sum is $\frac{1}{12}n+\frac{1}{4}n^2$.

Ex. 14. What is the nth term, and sum of n terms of the series $\frac{n-1}{n}, \frac{n-2}{n}$, &c.?

Ans. nth term $=0$, sum $=\frac{n-1}{2}$.

Ex. 15. The first term is $n^2-(n-1)$ and common difference is 2. What is the sum of n terms?
Ans. n^3.

From this example it follows that $3+5=2^3$, $7+9+11=3^3$, $13+15+17+19=4^3$, $21+23+25+27+29=5^3$, &c.

SECTION III.

Geometrical Progression.

60. A series of numbers increasing by a common multiplier, or decreasing by a common divisor, is said to be in *geometrical progression*. See Art. 42.

Thus, the series 2, 4, 8, 16,
and also 729, 243, 81, 27,

are in geometrical progression, the former being an increasing and the latter a decreasing series.

In general the series

$$a, ar, ar^2, ar^3, \&c.,$$

will represent any geometrical progression, as r may be taken integral or fractional.

61. The common multiplier r is called the *ratio*, which is a proper fraction if the series is decreasing.

62. The nth term of such a series is evidently of the form

$$l = ar^{n-1}.$$

63. To find the sum of n terms of the series, assume

$$S = ar^{n-1} + ar^{n-2} \dots\dots ar^2 + ar + a,$$

multiply by $r-1$, and this will become,

$$(r-1)\,S = ar^n - a = a\,(r^n - 1).$$

$$\therefore \quad S = \frac{a(r^n-1)}{r-1} = \frac{rl-a}{r-1}. \qquad \text{(A)}$$

64. If the series is decreasing, r is a proper fraction, and we shall have

$$S = \frac{a(1-r^n)}{1-r} = \frac{a-rl}{1-r}. \qquad \text{(B)}$$

65. These formulæ give the following rule for summing a geometrical progression.

Raise the ratio to a power indicated by the number of terms; divide the difference between this power and unity, by the difference between the ratio and unity, and multiply the quotient by the first term.

66. To find the ratio we have

$$r^{n-1} = \frac{l}{a},$$

or $$r = \sqrt[n-1]{\frac{l}{a}}. \quad \text{(See Art. 88.)}$$

Also $$(r - 1)\, S = ar^n - a.$$

Whence $$ar^n - S\, r = a - S.$$

This last equation can only be solved in particular cases.

67. If a decreasing series continue to infinity, the last term is 0, and the formula (B), Art. 64, becomes

$$S = \frac{a}{1 - r}. \qquad \text{(C)}$$

EXAMPLES.

Ex. 1. What is the sum of the first ten terms of the series 3, 6, 12, &c.

Here $a = 3$, $r = 2$, and $n = 10$,

$\therefore$ $r^n = 2^{10} = 1024$,

and $$S = a \cdot \frac{r^n - 1}{r - 1} = 3 \times 1023 = 3069.$$

Ex. 2. The first term is 78732, the number of terms 8, and the ratio $\frac{1}{3}$. What is the sum of the series?

Here $$l = ar^{n-1} = 78732 \times \frac{1}{2187} = 36,$$

and (B) $$S = \frac{a - rl}{1 - r} = \frac{78732 - 12}{\frac{2}{3}} = 118080.$$

Ex. 3. What is the sum of the series 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, ad infinitum?

Here $r = \frac{1}{3}$. Hence (C) $S = \dfrac{1}{1 - \frac{1}{3}} = \frac{3}{2}$.

Ex. 4. Insert 2 geometric means between 7 and 3584.

As there are 2 means there will be four terms.

Hence (66) $r = \sqrt[3]{\frac{l}{a}} = \sqrt[3]{\frac{3584}{7}} = \sqrt[3]{512} = 8$,

and the means are 56 and 448.

Ex. 5. What is the sum of the series $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27}$, ad infinitum?

Here $r = -\frac{1}{3}$ and $S = \frac{3}{4}$.

Ex. 6. Required the sum of 11 terms of the progression 3, 9, 27, 81. . . . *Ans.* 265719.

Ex. 7. Required the sum of the series 1, $\frac{1}{3}$, $\frac{1}{9}$, &c., to 15 terms. *Ans.* $1\frac{2391484}{4782969}$.

Ex. 8. A person wishing to purchase a fine horse was told he might have him, if he would give 1 mill for the first nail in his shoes, 2 for the second, 4 for the third, and so on. What would be the price of the horse at that rate, there being 8 nails in each shoe? *Ans.* $4294967.295.

Ex. 9. Required the sum of the series $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64}$, &c., ad infinitum. *Ans.* $\frac{4}{5}$.

Ex. 10. Required the sum of the first ten terms of the series 1, $\frac{1}{3}$, $\frac{1}{9}$, &c. *Ans.* $1\frac{9841}{19683}$.

Ex. 11. Insert four geometric means between 9 and 9216. *Ans.* 36, 144, 576, and 2304.

Ex. 12. What is the sum of $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81}$, &c., ad infinitum? *Ans.* $\frac{3}{4}$.

Ex. 13. What is the sum of $1 + \frac{1}{2} + \frac{1}{4} +$ &c., ad infinitum? *Ans.* 2.

Ex. 14. What is the sum of the series 100, 40, 16, &c., ad infinitum? *Ans.* $166\frac{2}{3}$.

Ex. 15. What is the sum of $x^{\frac{5}{2}} - bx + \frac{b^2}{\sqrt{x}}$ — &c., continued to infinity? (See Art. 89.) *Ans.* $\frac{x^4}{x^{\frac{3}{2}} + b}$.

Ex. 16. Insert three geometric means between $\frac{1}{9}$ and 9. *Ans.* $\frac{1}{3}$, 1, and 3.

Ex. 17. Insert three geometric means betwen 39 and 3159, also four between $\frac{1}{2}$ and 512.
Ans. 117, 351, and 1053; and 2, 8, 32, and 128.

Ex. 18. What is the sum of n terms of the series $a, b, \frac{b^2}{a}, \frac{b^3}{a^2}$, &c.; and also of the same series continued to infinity?
Ans. Sum of the terms $\frac{a^n - b^n}{(a-b)\,a^{n-2}}$; to inf. $\frac{a^2}{a-b}$.

SECTION IV.

Harmonical Proportion.

68. Three quantities are said to be in *harmonical* proportion, if the first is to the third as the difference between the first and second is to the difference between the second and third.

Thus, if $a : c :: a - b : b - c$; the magnitudes a, b, and c are in harmonical proportion.

69. Four quantities are in harmonical proportion where the first is to the fourth as the difference between the first and second is to the difference between the third and fourth.

Thus, if $a : d :: a - b : c - d$, a, b, c, and d are in harmonical proportion.

70. A harmonical *progression* is a series, any three consecutive terms of which are in harmonical proportion.

71. Let a, b, and c be three quantities in harmonical proportion,

then $a : c :: a - b : b - c$,

or (Art. 43,) $ab - ac = ac - bc$,

$\therefore$ $ab + bc = 2\,ac$,

whence $b = \frac{2\,ac}{a + c}$.

72. The reciprocals of any series in harmonical progression are in arithmetical progression.

Let a, b, c, d, e, &c., be in harmonical progression,

then will $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}$, &c., be in arithmetical progression,

for we will have $b = \frac{2\,ac}{a + c}$, $c = \frac{2\,bd}{b + d}$, $d = \frac{2\,ce}{c + e}$, &c.,

dividing by 2 and inverting we have

$$\frac{2}{b} = \frac{a + c}{ac} = \frac{1}{a} + \frac{1}{c}$$

$$\frac{2}{c} = \frac{b + d}{bd} = \frac{1}{b} + \frac{1}{d}$$

$$\frac{2}{d} = \frac{c + e}{ce} = \frac{1}{c} + \frac{1}{e},$$

therefore $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, &c., are in arithmetical progression. (58)

Thus the numbers $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, (or bringing to a common denominator, and using the numerators) 30, 20, 15, 12, 10, are in harmonical progression.

NOTE.—When five strings of equal weight and tension have their lengths as the above numbers, they will vibrate so as to make the most perfect harmony they can produce. Hence the name *Harmonical progression.*

73. *To find a harmonical mean between two quantities a and c.*

If b be the mean, we must have (Art. 71)

$$b = \frac{2\,ac}{a + c}$$

74. We may, by article 72, insert any number of harmonical means between two given numbers; it being only necessary to find the same number of arithmetical means between their reciprocals, and then take the reciprocals of the results.

EXAMPLES.

Ex. 1. Find the harmonical mean between 15 and 35.

Here the reciprocals are $\frac{1}{15}$ and $\frac{1}{35}$.

Their half sum being $\frac{1}{21}$ the required mean is 21,

Or, (73) $m = \frac{2ab}{a+b} = \frac{1050}{50} = 21.$

Ex. 2. Insert three harmonic means between 35 and 315.

Here the reciprocals are $\frac{1}{35}$ and $\frac{1}{315}$, or $\frac{9}{315}$ and $\frac{1}{315}$,

the three arithmetic means are, consequently,

$$\frac{7}{315}, \frac{5}{315} \text{ and } \frac{3}{315} = \frac{1}{45}, \frac{1}{63} \text{ and } \frac{1}{105},$$

and the harmonic means are

$$45, 63 \text{ and } 105.$$

Ex. 3. What is the harmonic mean between 119 and 170?
Ans. 140.

Ex. 4. What is the harmonic mean between 75 and 93?
Ans. $83\frac{1}{28}$.

Ex. 5. Insert two harmonic means between 15 and 81.
Ans. $20\frac{35}{59}$, and $32\frac{31}{37}$.

Ex. 6. Two consecutive terms of a harmonic progression are 5 and 9. Continue the series. *Ans.* 45.

and proceeding backwards; $3\frac{6}{13}$, $2\frac{11}{17}$, $2\frac{1}{7}$, $1\frac{4}{5}$, &c., 45 being the largest number of the series.

Ex. 7. Continue the harmonic series, two consecutive terms being 21 and 60. *Ans.* 60, 21, $12\frac{8}{11}$, $9\frac{3}{23}$, $7\frac{7}{59}$, &c.

SECTION V.

Permutations and Combination.

75. It is evident that two quantities, as a and b, may be arranged in two ways, viz., ab and ba. If there are three, a, b and c, they admit of being placed in six different manners, viz., abc, acb, bac, bca, cab and cba.

These different arrangements are called *Permutations*. The determination of their number is the object of the present section.

76. We have seen that two quantities, a and b, may be arranged in two orders, either being placed first. If we have three, a, b, c, either of these may be written first, and the remaining two arranged in two orders, so that we shall have 2.3 for the number of permutations of three quantities. Thus we have

$$\begin{array}{lll} a\ bc, & b\ ac & c\ ab \\ a\ cb & b\ ca & c\ ba. \end{array}$$

If we have four quantities; either being written first, the number of permutations of the remaining three is 2. 3. Hence the whole number of permutations of the four quantities is $2.\ 3.\ 4 = 24$.

Thus,

$$\begin{array}{llll} a\ bcd & b\ acd & c\ abd & d\ abc \\ a\ bdc & b\ adc & c\ adb & d\ acb \\ a\ cbd & b\ cad & c\ bad & d\ bac \\ a\ cdb & b\ cda & c\ bda & d\ bca \\ a\ dbc & b\ dac & c\ dab & d\ cab \\ a\ dcb & b\ dca & c\ dba & d\ cba \end{array}$$

In like manner we shall find the number of permutations of five numbers to be $2.\ 3.\ 4.\ 5 = 120$,

and so on, as in the following table,

Number of permutations of	2	quantities	$= 2$
	3	"	$= 2.3$
	4	"	$= 2.3.4$
	5	"	$= 2.3.4.5$
	:		
	n	"	$= 2.3.4.5 \ldots\ldots n.$

EXAMPLES.

Ex. 1. Required the number of orders in which five persons can arrange themselves at table? *Ans.* 120.

Ex. 2. How long will it require ten persons to arrange themselves in all possible orders, provided it requires five minutes to make every change, reckoning twelve hours per day? *Ans.* 69 years, and 15 days.

Ex. 3. How many numbers, each containing nine figures, all different, can be written with the nine digits 1 . . . . . 9?
Ans. 362880.

77. In the preceding article, we have determined the number of permutations of n quantities, using the whole number each time. In the present we will show how we may obtain the number when taken p at a time.

Let a, b, c, d . . . . . . be the quantities. If they are taken singly, the whole number of permutations is evidently n.

Now, since each of the n quantities may be placed before every one of the remaining $n-1$ quantities, we have for the number of permutations of n quantities taken two at a time,

$$n.(n-1).$$

Again, each of these $n.(n-1)$ arrangements of two at a time may be placed before every one of the remaining $n-2$ quantities; so that we shall have

$$n.(n-1).(n-2)$$

permutations of n quantities taken 3 at a time.

Proceeding in this manner we shall find that *the number of permutations of n quantities taken*

2 at a time is $n.(n-1)$
3 " " $n\ (n-1).(n-2)$
4 " " $n.(n-1).(n-2).(n-3)$
:
:
p " " $n.(n-1).(n-2).(n-3) \ldots\ldots (n-p+1)$
: " "
n " " $n.(n-1).(n-2).(n-3) \ldots\ldots 2.1$

which last result evidently agrees with last article.

78. In article 76 we have supposed the quantities to be all different. If, however, a be repeated twice, every arrangement will be found twice. Thus, if a and b be equal, the arrangement

$$a\ c\ b\ d, \text{ \&c.}$$

is the same as

$$b\ c\ a\ d, \text{ \&c.}$$

hence the number of different permutations is

$$\frac{n\,.\,(n-1)\ldots\ldots 2\,.\,1}{2}.$$

If one of the quantities be repeated 3 times, the number will be

$$\frac{n\,.\,(n-1)\ldots . 2\,.\,1}{2\,.\,3.}.$$

In like manner, if one quantity be repeated a times, another b times, and a third c times. The number of permutations will be

$$\frac{n\,.\,(n-1).(n-2)\ldots\ldots 2.1}{1.2\ldots . a\,.\,1.2\ldots . b.\,1.2\ldots . c}.$$

Examples.

Ex. 1. How many permutations of 5 quantities may be made from 10 different ones?

Ans. $10 \times 9 \times 8 \times 7 \times 6 = 30240$.

Ex. 2. How many different numbers, each containing four figures, may be formed of the nine significant digits?

Ans. 3024.

Ex. 3. In how many different manners may five letters be selected from the alphabet? *Ans.* 7893600.

Ex. 4. In how many ways may the figures of the number 36376 be arranged? *Ans.* 30.

Ex. 5. How many ways may the letters in the word Philadelphia be arranged? *Ans.* 14968800.

Ex. 6. How many numbers, each of six figures, may be formed of the digits in 363136? *Ans.* 60.

79. The *combinations* of any number of quantities taken p at a time, are the number of selections that can be made from these quantities, irrespective of the order in which they are placed.

Thus *abc*, *acb*, *bac*, *bca*, *cab* and *cba* form six permutations, but only one combination.

80. Now, since the number of permutations of n quantities, taken p at a time, is (Art. 77)

$$n\,.\,(n-1)\ldots\ldots(n-p+1),$$

and the number of permutations of p quantities amongst themselves is, (Art. 77)

$$p\,.\,(p-1)\ldots\ldots1;$$

the number of combinations of n quantities, taken p at a time, is

$$\frac{n\,.\,(n-1)\ldots\ldots(n-p+1)}{p\,.\,(p-1)\ldots\ldots1}.$$

Examples.

Ex. 1. In how many different ways may a flock of ten sheep be selected from a drove of 50?

Ans. $\frac{50.49.48.47.46.45.44.43.42.41}{1.2.3.4.5.6.7.8.9.10} = 10272278170.$

Ex. 2. In how many ways can five steers be selected from a drove of thirty? *Ans.* 142506.

Ex. 3. How many different combinations of seven letters may be made from the alphabet? *Ans.* 657800.

CHAPTER IV.

INVOLUTION AND EVOLUTION.

81. Involution is the process by which we determine the power of any number. As the power consists of the product of a number of factors, each equal to the root, it is plain that its value may be determined by multiplication; but as the process becomes tedious when the index of the power is large, it is more convenient to employ the following formula for the purpose. This formula, which is called the *binomial theorem*, was discovered by Sir Isaac Newton, who *appears* to have arrived at it by induction, as he has left no demonstration of its truth.

Let $x + a$ be any binomial. Then

$$(x+a)^n = x^n + nx^{n-1}a + n.\frac{n-1}{2}x^{n-2}a^2 + n\frac{n-1}{2}.\frac{n-2}{3}x^{n-3}a^3, \text{ \&c.}$$

To investigate this formula, we shall consider the product of the factors

$$(x+a).(x+b).(x+c).(x+d), \text{ \&c.}$$

First, $(x+a).(x+b) = x^2 + \begin{array}{c|} a \\ b \end{array} \, x + ab.$

$$(x+a).(x+b).(x+c) = x^3 + \begin{array}{c|} a \\ b \\ c \end{array} \, x^2 + \begin{array}{c|} ab \\ ac \\ bc \end{array} \, x + abc.$$

$$(x+a).(x+b).(x+c).(x+d)$$
$$= x^4 + \begin{array}{c|} a \\ b \\ c \\ d \\ \\ \\ \end{array} \, x^3 + \begin{array}{c|} ab \\ ac \\ ad \\ bc \\ bd \\ cd \end{array} \, x^2 + \begin{array}{c|} abc \\ abd \\ bcd \\ acd \\ \\ \\ \end{array} \, x + abcd.$$

In which the coefficient of the second term is the sum of the quantities a, b, c, d, &c. The coefficient of the third term is the sum of the products of the same quantities taken two at a time; that of the fourth is the sum of the products of the same quantities taken three at a time, and so on

82. We might from analogy conclude that this would always be the case; but that there may be nothing arbitrarily assumed, suppose the fact has been proven for n factors, we shall prove that it must likewise be true for $n+1$ factors.

Thus, let

$$(x+a)(x+b)(x+c)\dots\dots(x+n)$$

$$= x^n + \left.\begin{array}{c} a\\ b\\ c\\ \vdots\\ \vdots\\ n\\ \end{array}\right| x^{n-1} + \left.\begin{array}{c} ab\\ ac\\ ad\\ \vdots\\ \vdots\\ an\\ \&c. \end{array}\right| x^{n-2} + \left.\begin{array}{c} abc\\ abd\\ \vdots\\ \vdots\\ \vdots\\ abn\\ \&c. \end{array}\right| x^{n-3} + \dots abcd\dots n.$$

If we multiply this by another factor, we shall have

$$(x+a)(x+b)\dots\dots(x+n)(x+p) =$$

$$x^{n+1} + \left.\begin{array}{c} a\\ b\\ c\\ d\\ \vdots\\ n\\ \end{array}\right| x^n + \left.\begin{array}{c} ab\\ ac\\ ad\\ \vdots\\ \vdots\\ an\\ \&c. \end{array}\right| x^{n-1} + \left.\begin{array}{c} abc\\ abd\\ \vdots\\ \vdots\\ \vdots\\ abn\\ \&c. \end{array}\right| x^{n-2} + \dots abcd\dots nx.$$

$$+ px^n + \left.\begin{array}{c} ap\\ bp\\ cp\\ dp\\ \vdots\\ np\\ \end{array}\right| x^{n-1} + \left.\begin{array}{c} abp\\ acp\\ adp\\ \vdots\\ \vdots\\ anp\\ \&c. \end{array}\right| x^{n-2} \dots + abcd\dots np.$$

Now the second coefficient is evidently the sum of the $n+1$ quantities $a, b, c, \dots\dots p$; and if we examine the third we shall perceive that it is formed of, 1st, the sum of the products of the n quantities, taken 2 at a time; and, 2d, of all those combinations, 2 at a time, in which the new quantity p can enter.

The fourth coefficient consists, 1st, of all the combinations of the n quantities, 3 at a time; and, 2d, of all their combinations, 2 at a time, united with the new quantity p. Consequently, it is composed of the sum of the products of the p quantities taken three at a time, and so for the subsequent coefficients. Hence the law holds good.

83. This being admitted, it is evident (Art. 77) that the number of terms

in the second coefficient is n,

in the third " $\frac{n.(n-1)}{2}$,

in the fourth " $\frac{n.(n-1)(n-2)}{2.3}$,

in the fifth " $\frac{n.(n-1)(n-2)(n-3)}{2.3.4.}$, &c.

84. If we suppose t e quantities a, b, c, d, &c., to be all equal, ab, ac, &c., is each equal to a^2,

abc, adc, &c., " a^3,

and we shall have

$$(x+a)^n = x^n + nax^{n-1} + n.\frac{n-1}{2}a^2x^{n-2} + \frac{n.(n-1)(n-2)}{2.3}a^3x^{n-3}, \text{ \&c.}$$

The above demonstration is evidently confined to the case in which n is an integer. This being as much as was required in this part of algebra, it was thought proper to leave the more general demonstration to be given in the second part, which see.

85. From the formula of last article we obtain the following rule for involving a binomial.

The first term of the power is the first term of the root involved to the given power.

The literal parts of the succeeding terms consist of the successive powers of the first term of the root, regularly descending, joined to the successive powers of the second term, regularly ascending.

The coefficient of the second term is the index of the given power. That of the third term is obtained by mul-

tiplying the coefficient of the second term by the index of that power of the first term of the root contained in said term, and dividing by 2.

And in general the coefficient of any term is found by multiplying the coefficient and index in the preceding term, and dividing by the number of terms to that place.

EXAMPLES.

Ex. 1. Raise $a+x$ to the 5th power.

Here $(a+x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$.

$$\begin{array}{ccc} 4 & 3 & 2 \\ 2\,)\,\overline{20} & 3\,)\,\overline{30} & 4\,)\,\overline{20} \\ \overline{10} & \overline{10} & \overline{5} \end{array}$$

Ex. 2.

$(a-x)^6 = a^6 - 6a^5x + 15a^4x^2 - 20a^3x^3 + 15a^2x^4 - 6ax^5 + x^6$.

$$\begin{array}{cc} 5 & 4 \\ 2\,)\,\overline{30} & 3\,)\,\overline{60} \\ \overline{15} & \overline{20} \end{array}$$

Ex. 3. $(b-2x)^7 = b^7 - 7b^6(2x) + 21b^5(2x)^2 - 35b^4(2x)^3$
$+ 35b^3(2x)^4 - 21b^2(2x)^5 + 7b(2x)^6 - (2x)^7$
$= b^7 - 14b^6x + 84b^5x^2 - 280b^4x^3 + 560b^3x^4 - 672b^2x^5$
$+ 448bx^6 - 128x^7$.

Ex. 4. Raise $a-x$ to the 4th power.

Ex. 5. Raise $a-b$ to the 9th power.

Ex. 6. Raise $2-x$ to the 5th power.

Ex. 7. Raise $2a+3x$ to the 4th power.

Ex. 8. Raise $3x+y$ to the 5th power.

86. The principle contained in last article may readily be extended to a trinomial. Thus,

Let it be required to find the 4th power of $a+b-c$.

Considering $b-c$ as a single quantity, we have
$(a+b-c)^4 = (a+(b-c))^4 = a^4 + 4a^3(b-c) + 6a^2(b-c)^2$
$+ 4a(b-c)^3 + (b-c)^4$, and

$$\left.\begin{array}{l} a^4 \\ +4a^3(b-c) \\ +6a^2(b-c)^2 \\ +4a(b-c)^3 \\ +(b-c)^4 \end{array}\right\} = \left\{\begin{array}{l} a^4 \\ +4a^3b-4a^3c \\ +6a^2b^2-12a^2bc+6a^2c^2 \\ +4ab^3-12ab^2c+12abc^2-4ac^3 \\ +b^4-4b^3c+6b^2c^2-4bc^3+c^4 \end{array}\right\}$$
$$= (a+b-c)^4.$$

Examples.

Ex. 1. Raise $a-b+c$ to the 3d power.

Ans. $\left\{\begin{array}{l} a^3-3a^2b+3ab^2-b^3+3a^2c-6abc \\ +3ac^2+3b^2c-3bc^2+c^3. \end{array}\right.$

Ex. 2. Raise $2a-x+y$ to the 3d power.

Ex. 3. Raise $x-y-z$ to the 4th power.

Ex. 4. Raise $2a-3b+c$ to the 3d power.

Ex. 5. What is the cube of $x-2y+z$?

Ex. 6. What is the 4th power of $4a+2b-c$?

Ex. 7. What is the 3d power of $2x^2-3x+4$?

Ex. 8. What is the 4th power of $1-2a+x^2$?

Ex. 9. What is the 3d power of $a^2-6ax+9x^2$?

SECTION II.

Evolution.

87. Evolution is the extraction of roots, or the determining of a number which, multiplied by itself a given number of times, will produce the proposed number.

The square root of a number, is a number which, being squared, will produce the proposed number.

The cube root of a number, is one which, being cubed, will produce the proposed number.

As evolution is the converse of involution, the rules for performing it will readily be derived from those of the latter rule.

In raising a number to a given power, we multiply its index by that of the power to which it is to be raised. Thus

$(a^2)^5 = a^{2\times5} = a^{10}$. Consequently, the fifth root of a^{10} is $a^{\frac{10}{5}} = a^2$.

Again, the third power of $5a^2x^3$ is $125\ a^6x^9$, consequently, the cube root of $125\ a^6x^9$ is $5\ a^{\frac{6}{3}}x^{\frac{9}{3}} = 5\ a^2x^3$.

Hence, to obtain any root of a monomial, we first extract the root of the numeral part, and multiply it by that of the literal portion, determined by dividing the indices of the different quantities by the proper number.

88. *Def.* To indicate a root of a number, we prefix the sign $\surd$ with the index of the root upon it. Thus:

$\sqrt[2]{x}$ or $\sqrt{x}$ is the square root of x.
$\sqrt[3]{x}$ is the cube root of x.
$\sqrt[4]{x}$ is the fourth root of x.

When the root of a polynomial is to be expressed, the quantity must have a vinculum (———) placed over it, or be enclosed within brackets ( ).

Thus $\sqrt{a^2+x^2}$ or $\surd(a^2+x^2)$ represents the square root of a^2+x^2; while $\surd a^2+x^2$ signifies that the square root of a^2 is to be added to x^2; and is therefore exactly equivalent to $x^2+\surd a^2$.

89. It has been shown above that the index of the root is obtained by dividing that of the power by the number which expresses the root to be extracted. Thus

$$\surd x^4 \text{ is } x^{\frac{4}{2}} = x^2$$
$$\sqrt[3]{x^{15}} \text{ is } x^{\frac{15}{3}} = x^5.$$

This leads us to a notation for roots, of great use in algebra, for we see that $x^{\frac{4}{2}}$ is the square root of the fourth power of x.
" $x^{\frac{15}{3}}$ is the cube root of the fifteenth power of x.
the *denominator* of the fractional index expressing the *root*, and the *numerator* the *power*.

Thus $\surd a$ and $a^{\frac{1}{2}}$ are convertible expressions, both indicating the square root of a. In the same manner $\sqrt[3]{a^2}$ is the same as $a^{\frac{2}{3}}$, $\sqrt[5]{a^6}$ is the same as $a^{\frac{6}{5}}$ and in general $\sqrt[m]{x^n} = x^{\frac{n}{m}}$.

What is the square root of $49\,a^6b^4$?

Here $\sqrt{49\,a^6x^4} = \pm 7\,a^{\frac{6}{2}}x^{\frac{4}{2}} = \pm 7\,a^3x^2$.*

EXAMPLES.

Ex. 1. What is the square root of $81\,a^4x^6$?

Ans. $\pm 9\,a^2x^3$.

Ex. 2. What is the fourth root of $16\,a^8y^{12}$?

Ans. $\pm 2\,a^2y^3$.

Ex. 3. What is the fifth root of $1024\,x^5y^{15}$?

Ans. $4\,xy^3$.

Ex. 4. What is the cube root of $-729\,x^6y^{12}$?

Ans. $-9\,x^2y^4$.

Ex. 5. What is the cube root of $-\frac{8\,x^6y^9}{27\,a^6}$?

Ans. $-\frac{2\,x^2y^3}{3\,a^2}$.

Ex. 6. What is the square root of $9\,a^4x$?

Ans. $\pm 3\,a^2x^{\frac{1}{2}}$.

Ex. 7. What is the cube root of $343\,a^5x^6$? *Ans.* $7\,a^{\frac{5}{3}}x^2$.

Ex. 8. What is the fifth root of $243\,a^4x^{10}$?

Ex. 9. What is the fourth root of $625\,a^4x^8$?

Ex. 10. What is the cube root of $216\,a^6b^{12}$?

Ex. 11. What is the square root of $81\,a^8x^{10}$?

Ex. 12. Multiply $\sqrt{16\,a^6x^2}$ by $\sqrt[3]{8\,a^9x^6}$.

Ex. 13. Divide $\sqrt[5]{32\,a^4x^{10}}$ by $\sqrt{64\,x^8}$.

Ans. $\pm \frac{a^{\frac{4}{5}}}{4\,x^2}$ or $\frac{1}{4}\,a^{\frac{4}{5}}x^{-2}$.

---

* The square, or any even root of a number, may have either the plus or minus sign; but any odd root is affected by the sign of the power. The even root of a NEGATIVE number is *impossible*.—(See *Imaginary Quantities*.)

Ex. 14. Extract the cube root of $-27\,a^9x^{-5}$.

Ans. $-3\,a^3x^{-\frac{5}{3}}$.

Ex. 15. What is the value of $\sqrt[4]{81\,a^4x^{-8}b^2}$?

Ex. 16. Multiply $\sqrt[3]{-27\,a^6x^{-9}}$ by $\sqrt[4]{256\,a^{-8}\,x^{12}}$.

Ex. 17. What is the *m*th root of $a^3\,b^n$?

Ex. 18. What is the *n*th root of $a^m\,b^pc^q$?

SQUARE ROOT OF POLYNOMIALS.

90. If we examine the square of the binomial $a+x$, or $a^2+2\,ax+x^2$, we find it to consist of the square of the first term, twice the product of the two, and the square of the last term. If, then, we deduct the square of the first term of the root, the remainder, $2\,ax+x^2$ or $(2\,a+x)\,x$ is composed of the product of the second term, and twice the first term plus the second.

To find the first term of the root, therefore, we must *take the root of the first term of the power. The remainder being divided by twice the first term plus the second, will give for quotient the second term of the root.*

The whole process may be arranged as follows, viz.:

$$\begin{array}{r|l} & a^2+2\,ax+x^2\ (a+x \\ & \underline{a^2} \\ 2\,a+x\ ) & 2\,ax+x^2 \\ & \underline{2\,ax+x^2} \end{array}$$

The rule may be expressed thus:

Having arranged the terms commencing with the highest power of one of the quantities,

Take the square root of the first term for the first term of the root. Subtract its square from the given quantity, and set down the remainder, for a dividual.

Divide the first term of this dividual by twice the ascertained root for the next term, which place in the root and also in the divisor.

Multiply the divisor thus completed, by the last found term of the root, and subtract the product from the dividual, and so proceed.

The student cannot fail to notice the coincidence of this rule with the one ordinarily given in arithmetic; which rule is in fact derived from the algebraic formula.

EXAMPLES.

Ex. 1. What is the square root of

$$x^4-4x^3y+6x^2y^2-4xy^3+y^4.$$

$$
\begin{array}{lll}
 & x^4-4x^3y+6x^2y^2-4xy^3+y^4 & (x^2-2xy+y^2 \\
 & x^4 & \\
\hline
2x^2-2xy\,) & -4x^3y+6x^2y^2 & \\
 & -4x^3y+4x^2y^2 & \\
\hline
2x^2-4xy+y^2\,) & \qquad 2x^2y^2-4xy^3+y^4 & \\
 & \qquad 2x^2y^2-4xy^3+y^4. & \\
\hline
\end{array}
$$

Ex. 2. Extract the square root of $4a^2+20ax+25x^2$.

Ex. 3. Extract the square root of $4a^4-20a^3x+37a^2x^2-30ax^3+9x^4$.

Ex. 4. Extract the square root of $9x^4-12x^3-2x^2+4x+1$.

Ex. 5. Extract the square root of $9a^4-36a^3x+72ax^3+36x^4$.

Ex. 6. Extract the square root of

$$x^6-6ax^5+15a^2x^4-20a^3x^3+15a^4x^2-6a^5x+a^6.$$

Ex. 7. Extract the square root of

$$9x^6-12x^5+10x^4-28x^3+17x^2-8x+16.$$

Ex. 8. Extract the square root of

$$16a^2+8ab-8ac+b^2-2bc+c^2.$$

Ex. 9. Extract the square root of $1+x$.

Ans. $1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+$ &c.

Ex. 10. Extract the square root of $a^2 + 2b$.

Ans. $a + \frac{b}{a} - \frac{b^2}{2a^3} + \frac{b^3}{2a^5} - \frac{5b^4}{8a^7} + \&c.$

Ex. 11. What is the square root of

$$\frac{1}{4}x^6 - x^5 + \frac{5}{4}x^4 - \frac{5}{8}x^3 + \frac{5}{16}x^2 - \frac{1}{16}x + \frac{1}{64}?$$

Ex. 12. What is the square root of

$$\frac{1}{4}a^4 - \frac{1}{3}a^3b + \frac{10}{9}a^2b^2 - \frac{2}{3}ab^3 + b^4?$$

Ex. 13. What is the square root of 2 or 1 + 1?

Ans. $1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \&c.$

Extraction of the Cube Root.

91. The cube of $a + x$, being $a^3 + 3a^2x + 3ax^2 + x^3$; if we omit the first term a^3, the remainder, $3a^2x + 3ax^2 + x^3$, may be written $(3a^2 + 3ax + x^2) \times x$. Now, x being the second term of the root, the divisor must be $3a^2 + 3ax + x^2$, which consists of three times the square of the first term of the root, plus three times the product of the first and second terms, plus the square of the second term.

The following rule evidently includes the various steps to arrive at the root required, viz.:

RULE.

Arrange the terms as in the square root, and take the cube root of the first term for the first term of the root. Subtract the cube of this term from the given power, and the remainder will be the dividual.

Take three times the square of the ascertained root for a trial divisor, by which divide the first term of the dividual for the next term of the root.

Complete the divisor, by adding thereto three times the product of the two terms of the root, and the square of the last term.

Multiply and subtract, and so proceed until the operation is completed.

EXAMPLES.

Ex. 1. Extract the cube root of

$$x^6+6x^5-40x^3+96x-64.$$

$$\begin{array}{ll}
 & x^6+6x^5-40x^3+96x-64\,(x^2+2x-4 \\
 & x^6 \\
3x^4+6x^3+4x^2 &)+6x^5-40x^3 \\
\quad\quad 6x^3+8x^2 & +6x^5+12x^4+8x^3 \\
\hline
3x^4+12x^3+12x^2 &)-12x^4-48x^3+96x-64 \\
\quad -12x^2-24x+16 & -12x^4-48x^3+96x-64
\end{array}$$

The second trial divisor, $3x^4+12x^3+12x^2$, is equal to $3(x^2+2x)^2$; and is completed by adding $3\times-4(x^2+2x)+(-4)^2$. The complete divisor is therefore $3(x^2+2x)^2+3\times-4(x^2+2x)+(-4)^2$, and consequently is formed precisely as the preceding one, x^2+2x being considered the first term of the root.

The trial divisors subsequent to the first may be found without the trouble of squaring the ascertained root, by adding to the last complete divisor, the product and twice the square which were employed in completing said divisor. Thus in the above example $3x^4+12x^3+12x^2=3x^4+6x^3+4x^2+6x^3+8x^2$.

The above rule is identical with that employed in arithmetic, except that 30 times the product of the last figure, and those found before, is used in completing the divisor. This change is rendered necessary by the decimal notation employed in arithmetic.

Ex. 2. What is the cube root of

$$a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3ac^2+3bc^2+3b^2c+c^3?$$

The second trial divisor (see form, on opposite page) $3a^2+6ab+3b^2$ is equal to $3(a+b)^2$, but may be found by the directions contained in the observations appended to last example. The quantity added to complete it, viz: $3ac+3bc+c^2=3(a+b)c+c^2$.

$$a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3ac^2+3bc^2+3b^2c+c^3\,(a+b+c$$
$$a^3$$
$$3a^2+3ab+b^2)\quad 3a^2b+3ab^2+b^3$$
$$3ab+2b^2\quad 3a^2b+3ab^2+b^3$$
$$3a^2+6ab+3b^2+3ac+3bc+c^2)\quad 3a^2c+6abc+3ac^2+3b^2c+3bc^2+c^3$$
$$3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$$

Ex. 3. Extract the cube root of

$$27a^6-135a^5x+279a^4x^2-305a^3x^3+186a^2x^4-60ax^5+8x^6.$$

$$27a^6-135a^5x+279a^4x^2-305a^3x^3+186a^2x^4-60ax^5+8x^6\,(3a^2-5ax+2x^2$$
$$27a^6$$
$$27a^4-45a^3x+25a^2x^2)\quad -135a^5x+279a^4x^2-305a^3x^3$$
$$-45a^3x+50a^2x^2\quad -135a^5x+225a^4x^2-125a^3x^3$$
$$27a^4-90a^3x+75a^2x^2\quad 54a^4x^2-180a^3x^3+186a^2x^4-60ax^5+8x^6$$
$$18a^2x^2-30ax^3+4x^4)\quad 54a^4x^2-180a^3x^3+186a^2x^4-60ax^5+8x^6.$$

Ex. 4. Extract the cube root of

$$8a^3 + 12a^2b + 6ab^2 + b^3.$$

Ans. $2a + b$.

Ex. 5. Extract the cube root of

$$27a^6 - 54a^5x + 36a^4x^2 - 8a^3x^3.$$

Ans. $3a^2 - 2ax$.

Ex. 6. Extract the cube root of

$$8x^6 - 36x^5 + 114x^4 - 207x^3 + 285x^2 - 225x + 125.$$

Ans. $2x^2 - 3x + 5$.

Ex. 7. Extract the cube root of

$8a^3 - 36a^2b + 12a^2c + 54ab^2 - 27b^3 - 36abc + 27b^2c + 6ac^2 - 9bc^2 + c^3$. *Ans.* $2a - 3b + c$.

Ex. 8. Extract the cube root of

$$\frac{1}{8}a^6 - \frac{1}{2}a^5 + \frac{41}{48}a^4 - \frac{43}{54}a^3 + \frac{41}{96}a^2 - \frac{1}{8}a + \frac{1}{64}.$$

Ans. $\frac{1}{2}a^2 - \frac{2}{3}a + \frac{1}{4}$.

Ex. 9. Extract the cube root of

$$\frac{8}{27}x^6 - x^5y + \frac{59}{24}x^4y^2 - \frac{219}{64}x^3y^3 + \frac{59}{16}x^2y^4 - \frac{9}{4}xy^5 + y^6.$$

92. *General Rule for Extracting Roots of a complete power.*

It has been shown (Art. 84)

$$(a + x)^n = a^n + na^{n-1}x + \&c. \qquad \text{(A)}$$

And, therefore,

$$\sqrt[n]{(a^n + na^{n-1}x + \&c.)} = a + x.$$

From this formula, the rule is readily derived. For, the first term of the root is the nth root of the first term of the power; and the second term of the root is equal to the second term of the power, divided by na^{n-1}.

Also, by the same article, (84), we have

$$(a + x + y)^n = ((a + x) + y)^n$$
$$= (a + x)^n + n(a + x)^{n-1}y + \&c.$$

Whence

$$\sqrt[n]{\{(a+x)^n + n(a+x)^{n-1}y + \&c.\}} = a + x + y.$$

Having, then, determined the first two terms of the root, if the nth power of these terms be taken from the given power, the remainder is

$$n(a+x)^{n-1}y + \&c.$$

of which the first term consists of the remaining term y multiplied by $n(a+x)^{n-1}$.

In order, then, to determine y it will only be necessary to divide the leading term of the remainder by

$$n(a+x)^{n-1};$$

but as in determining the quotient, the first term only of the divisor is employed, it will be sufficient to divide the first term of the remainder by

$$na^{n-1}.$$

Having thus determined y, if

$$(a+x+y)^n$$

be subtracted from the given power, and the first term of the remainder be divided by

$$na^{n-1},$$

the next term of the root will be determined; and thus we may proceed until the operation is completed. These different processes are contained in the following

RULE FOR EXTRACTING ROOTS.

Arrange the terms as directed in division, extract the root of the first term of the power, and subtract its power from the given quantity.

For a divisor to be used in all the subsequent parts of the operation, raise the root already determined to a power whose index is one less than the number of the root to be extracted, and multiply by said number.

Divide the first term of the remainder by the divisor, the quotient will be the second term of the root.

Raise the root thus determined to the given power, and subtract from the given quantity.

Divide the first term of the remainder by the divisor for the third term of the root, and so proceed.

Ex. 1. Extract the fourth root of

$$x^8 + 8x^7 + 28x^6 + 56x^5 + 70x^4 + 56x^3 + 28x^2 + 8x + 1.$$

$$x^8 + 8x^7 + 28x^6 + 56x^5 + 70x^4 + 56x^3 + 28x^2 + 8x + 1 \quad (x^2 + 2x + 1$$

$$x^8$$

$$4 \cdot (x^2)^3 = 4x^6)\ 8x^7$$

$$(x^2 + 2x)^4 = x^8 + 8x^7 + 24x^6 + 32x^5 + 16x^4$$

$$4x^6) \qquad 4x^6$$

$$(x^2 + 2x + 1)^4 = x^8 + 8x^7 + 28x^6 + 56x^5 + 70x^4 + 56x^3 + 28x^2 + 8x + 1.$$

Ex. 2. What is the fifth root of $32\,a^5 - 80\,a^4x + 80\,a^3x^2 - 40\,a^2x^3 + 10\,ax^4 - x^5$. *Ans.* $2\,a - x$.

Ex. 3. Extract the fourth root of $81\,a^8 - 216\,a^7x + 324\,a^6x^2 - 312\,a^5x^3 + 214\,a^4x^4 - 104\,a^3x^5 + 36\,a^2x^6 - 8\,ax^7 + x^8$. *Ans.* $3\,a^2 - 2\,ax + x^2$.

Ex. 4. Extract the cube root of $8\,x^6 - 36\,x^5 + 66\,x^4 - 63\,x^3 + 33\,x^2 - 9\,x + 1$. *Ans.* $2\,x^2 - 3\,x + 1$.

The preceding rule may be applied to the extraction of roots of numbers. The operation, however, being very laborious, the rule given by Mr. Horner in the Philosophical Transactions for 1819, which is developed in the second part of this treatise, will be found much more convenient.

CHAPTER V.

SURDS AND IMAGINARY QUANTITIES.

SECTION I.

Surds.

93. Any expression indicating a root which cannot be expressed accurately is called a *surd*, or irrational quantity. Such are $\sqrt{5}$, $\sqrt[3]{9}$, $\sqrt{a}$, $\sqrt[4]{a^3}$, &c.

The operations upon surds are of great importance;* we shall therefore treat of them pretty fully in the following pages.

It is generally most convenient to express the root by the fractional index; since then, as will be shown, the multiplication and division of such quantities are performed as though they were rational.

94. To reduce surds to others having a common index.

Let $\sqrt[3]{x^2}$ and $\sqrt[4]{a^3}$ be two surd quantities; expressing them by a fractional index, they are written

$$x^{\frac{2}{3}} \text{ and } a^{\frac{3}{4}},$$

---

* A surd having the sign of the square root is called a *quadratic* surd, and with the sign of the cube root, a *cubic* surd, &c.

reducing the indices to a common denominator, they become

$$x^{\frac{3}{12}} = \sqrt[12]{x^3} \text{ and } a^{\frac{9}{12}} = \sqrt[12]{a^9}.$$

From the mode of operation employed here, the general rule may be derived.

Ex. 2. Let $\sqrt[3]{5}$ and $\sqrt{3}$ be the surds. They become as above

$$\sqrt[3]{5} = 5^{\frac{1}{3}} = 5^{\frac{2}{6}} = \sqrt[6]{5^2} = \sqrt[6]{25}$$

$$\sqrt{3} = 3^{\frac{1}{2}} = 3^{\frac{3}{6}} = \sqrt[6]{3^3} = \sqrt[6]{27}.$$

Ex. 3. Reduce $\sqrt[3]{5}$ and $\sqrt[5]{3}$ and $\sqrt[4]{6}$ to surds having a common index.

Ex. 4. Reduce $\sqrt[7]{a^3}$ and $\sqrt[4]{6}$ to surds having a common index.

Ex. 5. Reduce $\sqrt[9]{5}$ and $\sqrt[5]{3}$ to surds having a common index.

Ex. 6. Reduce $\sqrt[5]{a^4}$ and $\sqrt[3]{b^2}$ to surds having a common index.

Ex. 7. Reduce $5^{\frac{3}{4}}$ and $3^{\frac{2}{3}}$ to surds having a common index.

Ex. 8. Reduce $7^{\frac{3}{5}}$ and $4^{\frac{2}{3}}$ to surds having a common index.

95. A rational quantity may be reduced to the *form* of a surd by raising it to the proper power, and indicating the root required.

Thus $3 = \sqrt{9} = \sqrt[3]{27} = \sqrt[4]{81}$, &c.

$a = \sqrt{a^2} = \sqrt[3]{a^3} = \sqrt[4]{a^4}$, &c.

Ex. 1. Reduce 7 to a quadratic surd. *Ans.* $\sqrt{49}$.

Ex. 2. Reduce a^2 to a cubic surd.

Ex. 3. Reduce $5a^2x$ to a biquadratic surd. *Ans.* $\sqrt[4]{}$.

Ex. 4. Express $-7a$ in the form of the fifth root.

Ex. 5. Reduce $3a + 2$ to a quadratic surd.

Ex. 6. Reduce $5x + y$ to a cubic surd.

Ex. 7. Express $-3a^2y$ in the form of the fifth root.

96. It is of importance in operating with surds to be able to reduce them to their most simple form.

To do this, the fractional index must be reduced to its lowest terms, and the rational factors then separated from the others.

Thus, let $\sqrt[6]{54^2}$ be the surd. The fractional index $\frac{2}{6}=\frac{1}{3}$ $\therefore \sqrt[6]{54^2}=\sqrt[3]{54}$. Now, $54=27\times 2$, and 27 being a cube its root can be taken, while that of 2 is a surd. The result, therefore, will be $3\sqrt[3]{2}$.

The operation may be expressed as follows, viz.:

$$\sqrt[6]{54^2}=\sqrt[3]{54}=\sqrt[3]{27\times 2}=3\sqrt[3]{2}.$$

Ex. 2. Reduce $\sqrt[3]{16\,a^4b^3c^5}$ to its simplest form.

Here $8\,a^3b^3c^3$ is the greatest cube factor, hence

$$\sqrt[3]{16\,a^4b^3c^5}=\sqrt[3]{(8\,a^3b^3c^3\times 2\,ac^2)}=2\,abc\sqrt[3]{2\,ac^2}.$$

Ex. 3. Reduce $\sqrt{45}$ and $\sqrt[3]{500}$ to their simplest forms.

Ex. 4. Reduce $\sqrt[3]{192}$, $3\sqrt{245}$, $4\sqrt[5]{96}$ and $\sqrt{96}$ to their simplest forms.

Ex. 5. Reduce $x\sqrt{4\,a^2-8\,ax}$ and $\sqrt[3]{2\,a^4-5\,a^4x}$ to their simplest forms.

Ex. 6. Reduce $\sqrt{a^3\,(a^2-ax)}$ to its simplest form.

To reduce a fractional surd to its most simple form, its denominator must be made rational, which can always be done by the introduction of a suitable factor. Thus

Ex. 1. $\sqrt{\frac{1}{2}}=\sqrt{\frac{2}{4}}=\frac{\sqrt{2}}{2}=\frac{1}{2}\sqrt{2}.$

Ex. 2. $\sqrt{\frac{4}{5}}=\sqrt{\frac{20}{25}}=\sqrt{\left(\frac{4}{25}\times 5\right)}=\frac{2}{5}\sqrt{5}.$

Ex. 3. $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}=\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\times\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\frac{5+2\sqrt{6}}{3-2}$ $=5+2\sqrt{6}.$

Ex. 4. Reduce $\sqrt{\frac{3}{5}}$, $\sqrt{\frac{8}{3}}$, $\sqrt[3]{\frac{4}{9}}$, and $\sqrt{\frac{a^3b}{c}}$ to their simplest forms.

Ex. 5. Reduce $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ and $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$ to their simplest forms.

Ex. 6. Reduee $5\sqrt{\frac{3}{5}}$, $7\sqrt[3]{\frac{3}{7}}$, and $4\sqrt{\frac{3}{11}}$ to their simplest forms.

Ex. 7. Reduce $\frac{5}{\sqrt{2}}$, $\frac{7\sqrt[3]{3}}{\sqrt[3]{2}}$, and $\frac{5}{\sqrt{7}+\sqrt{3}}$ to their simplest forms.

Ex. 8. Reduce $\sqrt[3]{\frac{3}{7}}$, $\frac{6\sqrt[3]{5}}{\sqrt[3]{6}}$, and $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ to their simplest forms.

Ex. 9. Reduce $\frac{3+\sqrt{5}}{3-\sqrt{5}}$ and $\frac{7-\sqrt{3}}{7+\sqrt{3}}$ to their most simple forms.

97. To add or subtract surds, they should be reduced to their simplest forms, and the operations then performed as with rational quantities. Thus,

Ex. 1. Let the sum of $\sqrt{20}$ and $\sqrt{135}$ be required. The operation is as follows, viz.:

$$\sqrt{20} = \sqrt{(4\times 5)} = 2\sqrt{5}$$
$$\sqrt{125} = \sqrt{(25\times 5)} = 5\sqrt{5}$$

$\therefore$ the sum is $7\sqrt{5}$.

Ex. 2. Add $3\sqrt{45}$, $7\sqrt{20}$, $8\sqrt{125}$ and $2\sqrt{80}$ together.

Ex. 3. What is the sum of $3\sqrt{\frac{2}{3}}$ and $5\sqrt{\frac{3}{8}}$?

Ex. 4. Add $7\sqrt{5}+3\sqrt{27}$, $9\sqrt{20}-5\sqrt{75}$ and $2\sqrt{125}-\sqrt{12}$.

Ex. 5. From $3\sqrt{\frac{2}{5}}$ subtract $2\sqrt{\frac{5}{32}}$.

Ex. 6. From $7\sqrt[3]{189}$ subtract $3\sqrt[3]{168}$.

Ex. 7. Add $6\sqrt{27}$, $9\sqrt{192}$ and $7\sqrt{75}$.

Ex. 8. From $3\sqrt{\frac{1}{3}}+4\sqrt{\frac{3}{5}}$ subtract $\frac{5}{\sqrt{3}}-6\sqrt{135}$.

98. Multiplication and Division.

Let it be required to multiply $\sqrt[5]{a^2}$ by $\sqrt[3]{a}$.

These quantities expressed with fractional indices are

$$a^{\frac{2}{5}} \text{ and } a^{\frac{1}{3}}, \text{ or } a^{\frac{6}{15}} \text{ and } a^{\frac{5}{15}}.$$

The product of these evidently is

$$\sqrt[15]{a^{11}} \text{ or } a^{\frac{11}{15}}.$$

Hence, to multiply different roots of the same quantity, we add the fractional indices, and, of course, to divide different roots of the same quantity, we subtract the indices.

To make this matter, if possible, clearer, we must recollect, that the involution of monomials is performed by involving separately each of the simple quantities of which the monomial consists. Thus

$$(3\,ab^2)^5 = 3^5 \times a^5 \times (b^2)^5 = 243\,a^5b^{10}.$$

This being the case, the evolution of similar quantities must be performed by taking the root of each factor separately. Therefore,

$$\sqrt[15]{a^6} = \sqrt[15]{aaaaaa} = \sqrt[15]{a} \times \sqrt[15]{a} \times \sqrt[15]{a} \times \sqrt[15]{a} \times \sqrt[15]{a} \times \sqrt[15]{a},$$

and $\sqrt[15]{a^5} = \sqrt[15]{a} \times \sqrt[15]{a} \times \sqrt[15]{a} + \sqrt[15]{a} \times \sqrt[15]{a}.$

These, multiplied, evidently give $\sqrt[15]{a^{11}}$, as above.

99. In regard to the multiplication and division of roots of different quantities, the surds must first be reduced, so as to have a common index. The operation is then performed on the general principles of multiplication or division.

Examples.

Ex. 1. Multiply $7\sqrt[3]{5}$ by $4\sqrt{2}$.

Here $\quad 7\sqrt[3]{5} = 7\sqrt[6]{5^2} = 7\sqrt[6]{25}$

and $\quad 4\sqrt{2} = 4\sqrt[6]{2^3} = 4\sqrt[6]{8}$

∴ the product is $\quad 28\sqrt[6]{200}$.

Ex. 2. Divide $\frac{5}{3}\sqrt[3]{\frac{3}{8}}$ by $\frac{9}{10}\sqrt[3]{\frac{2}{5}}$.

Here, by the general principle of the division of fractions, we have

$$\frac{5}{3}\sqrt[3]{\frac{3}{8}}\div\frac{9}{10}\sqrt[3]{\frac{2}{5}}=\frac{5}{3}\sqrt[3]{\frac{3}{8}}\times\frac{10}{9}\sqrt[3]{\frac{5}{2}}=\frac{50}{27}\sqrt[3]{\frac{15}{16}}=\frac{50}{27}\sqrt[3]{\frac{60}{64}}$$

$$=\frac{50}{27}\sqrt[3]{\left(\frac{1}{64}\times60\right)}=\frac{50}{27}\times\frac{1}{4}\sqrt[3]{60}=\frac{25}{54}\sqrt[3]{60},$$

which is the quotient in its simplest form.

Ex. 3. Multiply $3\sqrt{17}$ by $5\sqrt{8}$.

Ex. 4. Multiply $5\sqrt{\frac{3}{5}}$ by $7\sqrt{\frac{4}{7}}$.

Ex. 5. Divide $3\sqrt{15}$ by $4\sqrt{5}$.

Ex. 6. Divide $\frac{5}{7}\sqrt{\frac{3}{4}}$ by $\frac{3}{5}\sqrt{\frac{6}{7}}$.

Ex. 7. Multiply $8\sqrt{3}$ and $9\sqrt[3]{7}$.

Ex. 8. Multiply $6\sqrt[3]{\frac{5}{6}}$ by $3\sqrt{\frac{2}{3}}$.

Ex. 9. Divide $\frac{1}{2}\sqrt[3]{\frac{3}{4}}$ by $\frac{2}{3}\sqrt{\frac{4}{5}}$.

Ex. 10. Multiply $3+\sqrt{5}$ by $2-\sqrt{5}$.
Ans. $1-\sqrt{5}$.

Ex. 11. Multiply $4+3\sqrt{7}$ by $2-2\sqrt{7}$.
Ans. $-34-2\sqrt{7}$.

Ex. 12. Multiply $\frac{1}{2}+\frac{1}{2}\sqrt{5}$ by $\frac{3}{4}+\frac{1}{4}\sqrt{5}$.
Ans. $1+\frac{1}{2}\sqrt{5}$.

Ex. 13. Multiply $\sqrt{3}+3\sqrt{2}$ by $\sqrt{2}+2\sqrt{3}$.
Ans. $12+7\sqrt{6}$.

Ex. 14. Multiply $3\sqrt{7}-2\sqrt{5}$ by $2\sqrt{7}+3\sqrt{5}$.
Ans. $5\sqrt{35}+12$.

Ex. 15. Multiply $3\sqrt{5}+7\sqrt{3}-3\sqrt{2}$ by $\sqrt{20}-2\sqrt{12}+\sqrt{8}$.
Ans. $2\sqrt{15}+26\sqrt{6}-66$.

Ex. 16. Divide 41 by $9 + 2\sqrt{10}$.

Ans. $9 - 2\sqrt{10}$.

This is most readily done by expressing the quotient as a fraction, and reducing to the simplest form, as in Art. 96.

Ex. 17. Divide $24 + 7\sqrt{10}$ by $3\sqrt{5} - \sqrt{2}$.

Ans. $2\sqrt{5} + 3\sqrt{2}$.

100. To extract the root of a monomial surd, we have only to apply the principles already explained, (Art. 87.)

EXAMPLES.

Ex. 1. The cube root of $125\sqrt{x}$ is required.

Here $\sqrt[3]{(125\sqrt{x})} = \sqrt[3]{(125\,x^{\frac{1}{2}})} = 5\,x^{\frac{1}{2}\div 3} = 5\,x^{\frac{1}{6}}$ $= 5\sqrt[6]{x}$.

Ex. 2. What is the square root of $81\,a^{\frac{2}{3}}b^{\frac{1}{4}}$?

Ex. 3. What is the cube root of $216\sqrt[4]{ab^3}$?

Ex. 4. What is the fifth root of $32\sqrt[3]{a^2b^5}$?

101. When the surd consists of two terms, one of which is rational and the other a quadratic surd, its root may sometimes be obtained by the following formula, viz.:

$$\sqrt{\left(a\pm\sqrt{b}\right)}=\sqrt{\left(\frac{1}{2}a+\frac{1}{2}\sqrt{a^2-b}\right)}\pm\sqrt{\left(\frac{1}{2}a-\frac{1}{2}\sqrt{a^2-b}\right)},$$

which may be demonstrated as follows:

Let $\sqrt{a+\sqrt{b}} = \sqrt{x} + \sqrt{y}$. Squaring both members of the equation, it becomes

$$a + \sqrt{b} = x + y + 2\sqrt{xy}.$$

And as a surd cannot be equal to a rational quantity, we must have

must have $\quad x + y = a \quad (1)$

and $\quad 2\sqrt{xy} = \sqrt{b},$

whence $\quad 4\,xy = b. \quad (2)$

If from the square of (1) we subtract (2), we will have

$$x^2 - 2\,xy + y^2 = a^2 - b,$$

whence extracting the square root of both members

$$x-y=\sqrt{a^2-b},$$

but $$x+y=a$$

$\therefore$ $$2x=a+\sqrt{a^2-b},$$

and $$2y=a-\sqrt{a^2-b},$$

or $$x=\frac{1}{2}a+\frac{1}{2}\sqrt{a^2-b},$$

$$y=\frac{1}{2}a-\frac{1}{2}\sqrt{a^2-b}.$$

$$\therefore \sqrt{(a+\sqrt{b})}=\sqrt{\left(\frac{1}{2}a+\frac{1}{2}\sqrt{a^2-b}\right)}+\sqrt{\left(\frac{1}{2}a-\frac{1}{2}\sqrt{a^2-b}\right)}$$

Again, since $a=x+y$, and $\sqrt{b}=2\sqrt{xy}$, we have

$$a-\sqrt{b}=x+y-2\sqrt{xy}.$$

Now the 2d member of this is evidently the square of

$$\sqrt{x}-\sqrt{y}.$$

$$\sqrt{(a-\sqrt{b})}=\sqrt{x}-\sqrt{y},$$

$$=\sqrt{\left(\frac{1}{2}a+\frac{1}{2}\sqrt{a^2-b}\right)}-\sqrt{\left(\frac{1}{2}a-\frac{1}{2}\sqrt{a^2-b}\right)}$$

In order that the result should appear in a simple form, it is evidently necessary that a^2-b should be a square.

EXAMPLES.

Ex. 1. Extract the square root of $7+4\sqrt{3}$.

Here $a=7$, and $\sqrt{b}=4\sqrt{3}$, $\therefore b=48$.

Hence

$$\sqrt{(7+4\sqrt{3})}=\sqrt{\left(\frac{7}{2}+\frac{1}{2}\sqrt{49-48}\right)}+\sqrt{\left(\frac{7}{2}-\frac{1}{2}\sqrt{49-48}\right)}$$

$$=\sqrt{\left(\frac{7}{2}+\frac{1}{2}\right)}+\sqrt{\left(\frac{7}{2}-\frac{1}{2}\right)}=2+\sqrt{3}.$$

Ex. 2. What is the square root of $5+\sqrt{24}$?

Ans. $\sqrt{2}+\sqrt{3}$.

Ex. 3. What is the square root of $7-\sqrt{13}$?

Ans. $\frac{1}{2}\sqrt{26}-\frac{1}{2}\sqrt{2}$

Ex. 4. What is the square root of $10 - \sqrt{51}$?
Ans. $\frac{1}{2}\sqrt{34} - \frac{1}{2}\sqrt{6}$.

Ex. 5. What is the square root of $7 + 2\sqrt{10}$?
Ans. $\sqrt{5} + \sqrt{2}$.

Ex. 6. What is the square root of $12 - 4\sqrt{5}$?
Ans. $\sqrt{10} - \sqrt{2}$.

Ex. 7. What is the square root of $37 + 12\sqrt{7}$?
Ans. $3 + 2\sqrt{7}$.

Ex. 8. What is the square root of $23 + 6\sqrt{10}$?
Ans. $3\sqrt{2} + \sqrt{5}$.

SECTION II.

Imaginary Quantities.

102. Every even power of a negative quantity being positive; it follows that such expressions as $\sqrt{-a^2}$, $\sqrt[4]{-b^4}$, &c., can have no real value. They are, therefore, called *Imaginary Quantities.*

Though imaginary quantities have no real value, yet as they are of much use in analysis, the principles on which the operations upon them are founded are of great importance.

103. In the algebraic solution of arithmetical questions, we never arrive at such results, unless the conditions of the problem are inconsistent. They therefore serve to point out such inconsistencies. For example, let the following question be proposed, viz.:

To divide a line, 8 yards in length, into two such parts that their rectangle may be 25.

If we represent the parts by x and y, the equations will be

$$x + y = 8,$$

and $$xy = 25.$$

From the 1st $x^2 + 2xy + y^2 = 64$.

But $$4xy = 100.$$

∴ subtracting this from the preceding, we have

$$x^2-2xy+y^2=-36.$$

And extracting the square root

$$x-y=\sqrt{-36}=6\sqrt{-1}.$$

Adding this equation to the first, and dividing by 2,

we have $\quad x=4+3\sqrt{-1},$

and $\quad y=4-3\sqrt{-1}.$

Since these values are imaginary, we conclude that the problem was absurd. This we know to be the case, since the greatest rectangle would be formed when the parts were equal. In this instance it is 16.

104. The addition and subtraction of imaginary quantities being performed on the same principles as the addition and subtraction of surds, present no difficulties; we shall, therefore, proceed to multiplication and division.

It is in the first place evident that

$$\sqrt{-a}\times\sqrt{-a}=-a;$$

so that in this case $\quad \sqrt{a^2}=-a.$

Now in general we may assert

that $\quad \sqrt{a^2}=+a, \text{ or } -a,$

which is written $\quad \pm a.$

It might from this be contended, that $+a=-a$. The reasoning, however, would be incorrect; for x is not equal to a and $-a$, but to either a or $-a$. In other words, there are two values to $\sqrt{a^2}$. These values are numerically equal, but of different signs. The symbol $=$ expresses, however, more than mere numerical equality, it implies perfect identity. We shall hereafter see many cases in which the required quantity admits of several values; but we cannot from thence conclude that these values are equal.

In the following problem, for example, we shall find that there are two numbers which satisfy the conditions; and consequently the letter which represents the unknown must have two values, viz.:

What number is that which, being subtracted from 10, and the remainder multiplied by the number itself, the product shall be 21?

Here, if x represent the number required, we shall have $x = 7$ or 3, as may be proven by trial. Thus,

$$(10-7) \times 7 = 3 \times 7 = 21, \text{and } (10-3) \times 3 = 7 \times 3 = 21.$$

It were folly thence to conclude that $7 = 3$.

So, although $\sqrt{a^2} = a$ or $-a$, it is not true that $a = -a$.

When, from the circumstances of the case, no means are afforded of determining the sign with which the root should be affected, the result is ambiguous. This produces no inconvenience when mere analytic operations are concerned. In fact, it is of advantage; for the formulæ would otherwise fail to present a full solution of the problem under consideration. In the application to arithmetical and geometrical problems, however, it sometimes happens that one of the results, though analytically correct, is excluded by the conditions of the problem. We shall see numerous examples of this in *Quadratic Equations.*

105. If the quantities be unequal, as $\sqrt{-a}$ and $\sqrt{-b}$, we may not be able at first to discover what should be the sign of their product $\sqrt{ab}$. When, however, we put them in the form $\sqrt{a} \cdot \sqrt{-1}$ and $\sqrt{b} \cdot \sqrt{-1}$, we readily perceive that their product is $\sqrt{ab} \times -1 = -\sqrt{ab}$.

From this, and the general principle of multiplication, we may form the following table, which includes the different cases of the multiplication of imaginary quantities.

$$\sqrt{-a} \times \sqrt{-a} = -a$$
$$-\sqrt{-a} \times -\sqrt{-a} = -a$$
$$-\sqrt{-a} \times \sqrt{-a} = +a$$
$$\sqrt{-a} \times \sqrt{-b} = -\sqrt{ab}$$
$$-\sqrt{-a} \times -\sqrt{-b} = -\sqrt{ab}$$
$$-\sqrt{-a} \times \sqrt{-b} = \sqrt{ab}$$
$$\sqrt{-a} \times -\sqrt{-b} = \sqrt{ab}.$$

Examples.

Ex. 1. Multiply $3\sqrt{-2}$ by $4\sqrt{-3}$.

Ans. $-12\sqrt{6}$.

Ex. 2. Multiply $6\sqrt{-5}$ by $-3\sqrt{-5}$. *Ans.* 90.

Ex. 3. Multiply $-7\sqrt{-3}$ by $-4\sqrt{-3}$.
Ans. -84.

Ex. 4. Multiply $-8\sqrt{-7}$ by $2\sqrt{-2}$.
Ans. $16\sqrt{14}$.

Ex. 5. Square $3\sqrt{-5}$. *Ans.* -45.

Ex. 6. Cube $-2\sqrt{-3}$. *Ans.* $+24\sqrt{-3}$.

Ex. 7. Multiply $3+2\sqrt{-3}$ by $4-3\sqrt{-3}$.

$$\begin{array}{l} 3+2\sqrt{-3} \\ 4-3\sqrt{-3} \\ \hline 12+8\sqrt{-3} \\ \quad -9\sqrt{-3}+18 \\ \hline 30-\sqrt{-3}. \end{array}$$

Ex. 8. Multiply $4-3\sqrt{-2}$ by $-2+6\sqrt{-2}$.
Ans. $28+30\sqrt{-2}$.

Ex. 9. Square $3\sqrt{-2}+2$. *Ans.* $-14+12\sqrt{-2}$.

Ex. 10. Cube $2-\sqrt{-3}$. *Ans.* $-10-9\sqrt{-3}$.

Ex. 11. Cube $a-b\sqrt{-1}$.
Ans. $a^3-3ab^2+(b^3-3a^2b)\sqrt{-1}$.

Ex. 12. Cube $a+b\sqrt{-1}$.
Ans. $a^3-3ab^2+(3a^2b-b^3)\sqrt{-1}$.

Ex. 13. Multiply $3\sqrt{-2}+5\sqrt{-3}$ by $2\sqrt{-2}-3\sqrt{-3}$. *Ans.* $33-\sqrt{6}$.

Ex. 14. Multiply $4\sqrt{-6}-3\sqrt{-5}$ by $2\sqrt{-6}+3\sqrt{-5}$. *Ans.* $-3-6\sqrt{30}$.

Ex. 15. Multiply $4\sqrt{2}-3\sqrt{-3}$ by $2\sqrt{3}+5\sqrt{-2}$.
Ans. $23\sqrt{6}+22\sqrt{-1}$.

106. Division of imaginary quantities is performed on the same principles as the division of surds. We shall therefore merely append a few examples for exercise.

Ex. 1. Divide $5\sqrt{-6}$ by $2\sqrt{-3}$.

$$\frac{5\sqrt{-6}}{2\sqrt{-3}}=\frac{5}{2}\sqrt{2}.$$

Ex. 2. Divide $2+\sqrt{-3}$ by $2-\sqrt{-3}$.

$\frac{2+\sqrt{-3}}{2-\sqrt{-3}}=$ (multiplying the terms by $2+\sqrt{-3}$) $\frac{1+4\sqrt{-3}}{7}$.

Ex. 3. Divide $-8\sqrt{-15}$ by $4\sqrt{-3}$.

Ans. $-\frac{3}{4}\sqrt{5}$.

Ex. 4. Divide $-7\sqrt{-6}$ by $-3\sqrt{-4}$.

Ans. $\frac{7}{3}\sqrt{\frac{3}{2}}=\frac{7}{6}\sqrt{6}$.

Ex. 5. Divide $3-\sqrt{-3}$ by $3+\sqrt{-3}$.

Ans. $\frac{1}{2}-\frac{1}{2}\sqrt{-3}$.

Ex. 6. Divide $1+\sqrt{-1}$ by $1-\sqrt{-1}$. *Ans.* $\sqrt{-1}$.

Ex. 7. Divide $5+\sqrt{-2}$ by $5-\sqrt{-2}$.

Ans. $\frac{23}{27}+\frac{10}{27}\sqrt{-2}$.

CHAPTER VI.

EQUATIONS.

107. An equation (as defined Art. 6) is an expression of equality between two quantities; the quantities considered as equal, being the members or sides of the equation.

108. A *simple* equation is one in which the unknown quantity does not rise above the first degree.

Thus, $2x + 3 = 5x + 7$; $ax + bx = n$,

are simple equations.

109. A *quadratic*, or equation of the *second degree*, is one in which the square of the unknown appears; as in

$$4x^2 - 5x = 20.$$

110. A *cubic*, or equation of the *third* degree, contains the cube of the unknown; as

$$3x^3 - 5x^2 + 2x = 80.$$

111. A *biquadratic*, or equation of the fourth degree, contains the fourth power of the unknown; as

$$4x^4 - 30x^3 + 5x^2 + 8x = 70.$$

112. Those which contain higher powers than the fourth are called equations of the fifth, sixth, &c., degrees, according as the highest power of the unknown is the fifth, sixth, &c.

113. A pure equation is one which contains but a single power of the unknown quantity. Thus,

$$ax^3 = b, \text{ and } 5x^4 = 80$$

are pure equations of the third and fourth degrees respectively.

114. Those equations which contain more than one power of the unknown quantity are called adfected equations.

115. A complete equation contains all the powers of the

unknown, from the highest down, and also a known term; thus,

$$x^4 - 5x^3 + 8x^2 + 3x + 20 = 0$$

is a complete equation of the fourth degree.

SECTION I.

Simple Equations.

Recapitulation of Rules.—(*Art.* 8 *and* 9.)

116. 1st. An equation may be cleared of fractions by multiplying by the least common multiple of the denominators.

2d. Any quantity may be transposed from one member of an equation to the other by changing its sign.

On these two rules depend all the methods of solving such simple equations as do not involve the roots of the unknown quantity. The artifices to be employed vary so much with the nature of the equation under consideration, that much must depend upon the ingenuity of the student. An equation should, however, generally be cleared of fractions; after which the terms containing the unknown quantity should be collected, by transposition, in the left member, and all the others in the right, then dividing by the coefficient of the unknown quantity, we arrive at the value desired.

EXAMPLES.

Ex. 1. Given $\frac{x+3}{7} - \frac{4}{5} = \frac{2(x-1)}{3} - \frac{9}{5}$ to find x.

Multiplying by 105, we have

$$15x + 45 - 84 = 70x - 70 - 189.$$

Transposing, $15x - 70x = -70 - 189 - 45 + 84,$

or, $-55x = -220,$

whence $x = 4.$

Ex. 2. Given $\frac{3x+4}{5} - \frac{22-x}{5} = 16 - 2x$, to find the value of x.

Clearing of fractions, we have

$$3x + 4 - 22 + x = 80 - 10x,$$

whence $$14x = 98,$$

and $$x = 7.$$

Ex. 3. Given $(ac + bd)^2 + b^2x = 2abcd + (a^2 + b^2)c^2$.

Squaring $ac + bd$ and multiplying the factors in the last term, we have

$$a^2c^2 + 2abcd + b^2d^2 + b^2x = 2abcd + a^2c^2 + b^2c^2$$

or omitting the terms found in both members

$$b^2d^2 + b^2x = b^2c^2,$$

whence $$b^2x = b^2c^2 - b^2d^2,$$

and $$x = c^2 - d^2.$$

Ex. 4. Given $\frac{x+5}{7} - 4\frac{3}{4} = 5x - \frac{3x-2}{8} + 7$, to find the value of x.

Clearing of fractions, we have

$$8x + 40 - 266 = 280x - 21x + 14 + 392,$$

or, $$-251x = 632$$

$$x = -2\frac{130}{251}.$$

Ex. 5. Given $7x + 6 - 3x = 56 + 2x$, to find the value of x. *Ans.* $x = 25$.

Ex. 6. Given $\frac{x}{2} + \frac{x}{3} + \frac{5}{6} = \frac{4x}{5} + 7$, to find the value of x. *Ans.* $x = 185$.

Ex. 7. Given $\frac{3x+6}{5} + 2x = \frac{17-3x}{2} - \frac{2x-31}{5}$, to find the value of x. *Ans.* $x = 3$.

Ex. 8. Given $\frac{2x-5}{x+3} - \frac{x}{4} = 15 - \frac{x+5}{4}$, to find the value of x. *Ans.* $x = -3\frac{44}{47}$.

Ex. 9. Given $\frac{2}{3}(x+8)-\frac{3}{4}(2x+7)=\frac{1}{2}(3x-5)$, to find the value of x. *Ans.* $x=1\frac{3}{28}$.

Ex. 10. Given $12x+7-\frac{(8x+9)(9x-7)}{6x-3}=-1\frac{13}{21}$ to find the value of x. *Ans.* $x=4$.

Ex. 11. Given $\frac{a}{bx}+\frac{b}{cx}=d$, to find the value of x.

Ans. $x=\frac{ac+b^2}{bcd}$.

Ex. 12. Given $(a+x)(a-x)-3ax=\frac{a^2-3x^2}{3}$, to find the value of x.

Ans. $x=\frac{2a}{9}$.

117. It very frequently happens that the unknown, either simply or in connection with known quantities, is affected with a radical sign. One of the first steps in such cases is to clear the equation of surds. This may be done readily if there is but one such quantity in the equation. For if we transpose so as to leave the radical stand by itself, and then involve the equation to a power indicated by the index of the surd, we shall evidently obtain a new equation clear of surds.

Thus, if the equation is

$$\sqrt{x^2-ax}+x=a+2x,$$

we have $$\sqrt{x^2-ax}=a+x.$$

Whence by squaring both numbers, we obtain

$$x^2-ax=a^2+2ax+x^2.$$

From this we find

$$x=-\frac{a}{3}.$$

118. If there are more than one surd in an equation, no general rules can be given as to the arrangement of the terms previously to involving. It depends in each case upon the nature of the surds; the method of proceeding must therefore be left to the ingenuity of the student.

In many instances the question will admit of various modes of solution.

EXAMPLES.

Ex. 1. Given $\sqrt{a+x} = \sqrt{ax} + \sqrt{a-x}$.

Squaring we have $a + x = ax + 2\sqrt{a^2x - ax^2} + a - x$,

whence $$2x - ax = 2\sqrt{a^2x - ax^2}.$$

Squaring $$4x^2 - 4ax^2 + a^2x^2 = 4a^2x - 4ax^2,$$

Cancelling $-4a^2x$, and dividing by x,

$$4x + a^2x = 4a^2$$

whence $$x = \frac{4a^2}{a^2+4}.$$

Another solution.

Transposing we have $\sqrt{a+x} - \sqrt{a-x} = \sqrt{ax}$,

Whence by squaring $a + x - 2\sqrt{a^2 - x^2} + a - x = ax$,

and transposing $-2\sqrt{a^2 - x^2} = ax - 2a$,

whence $$4a^2 - 4x^2 = a^2x^2 - 4a^2x + 4a^2,$$

transposing and dividing by x

$$4x + a^2x = 4a^2$$

$$\therefore \quad x = \frac{4a^2}{a^2+4}$$

Ex. 2. Given $\sqrt{x} + \sqrt{5+x} = \dfrac{10}{\sqrt{5+x}}$, to find the value of x.

Here clearing of fractions, we have

$$\sqrt{5x + x^2} + 5 + x = 10.$$

$$\therefore \quad \sqrt{5x + x^2} = 5 - x.$$

And $$5x + x^2 = 25 - 10x + x^2,$$

whence $$15x = 25$$

and $$x = \frac{5}{3}.$$

Ex. 3. Given $\sqrt{9+x} + 3 = 7$, to find the value of x.

Ans. $x = 7$.

Ex. 4. Given $\sqrt{a^2 + x^2} = b + x$, to find the value of x.

Ans. $x = \frac{a^2 - b^2}{2b}$.

Ex. 5. Given $\sqrt{a + x} + \sqrt{x} = \frac{2a}{\sqrt{a + x}}$, to find the value of x.

Ans. $x = \frac{a}{3}$.

Ex. 6. Given $\sqrt{a + x} = \sqrt[4]{4a^2 + x^2}$, to find the value of x.

Ans. $x = \frac{3a}{2}$.

Ex. 7. Given $a + x = \sqrt{\{a^2 + x\sqrt{(b^2 + x^2)}\}}$, to find the value of x.

Ans. $x = \frac{b^2 - 4a^2}{4a}$.

Ex. 8. Given $x + a + \sqrt{2ax + x^2} = b$, to find the value of x.

Ans. $x = \frac{(b - a)^2}{2b}$.

Ex. 9. Given $\frac{\sqrt{x} + 28}{\sqrt{x} + 4} = \frac{\sqrt{x} + 38}{\sqrt{x} + 6}$, to find the value of x.

Ans. $x = 4$.

Ex. 10. Given $\frac{\sqrt{ax} - b}{\sqrt{ax} + b} = \frac{3\sqrt{ax} - 2b}{3\sqrt{ax} + 5b}$, to find the value of x.

Ans. $x = \frac{9b^2}{a}$.

Ex. 11. Given $\sqrt{\{1 + x\sqrt{x^2 + 12}\}} = 1 + x$, to find the value of x.

Ans. $x = 2$.

Ex. 12. Given $\sqrt{2x - 4} + 12 = 14$, to find the value of x.

Ans. $x = 4$.

Ex. 13. Given $\sqrt[3]{a + x} + \sqrt[3]{a - x} = b$, to find the value of x.

Ans. $x = \sqrt{a^2 - \left(\frac{b^3 - 2a}{3b}\right)^3}$.

Ex. 14. Given $\sqrt[3]{6 + x} + \sqrt[3]{6 - x} = 2$, to find the value of x.

Ans. $x = \frac{14}{9}\sqrt{15}$.

Ex. 15. Given $\sqrt{a^2+x}+\sqrt{x}=\frac{2a^2}{\sqrt{(a^2+x)}}$, to find the value of x. *Ans.* $x=\frac{a^2}{3}$.

Ex. 16. Given $8x+6:5x+2::5:3$, to find the value of x. See Art. 43. *Ans.* $x=8$.

Ex. 17. Given $\frac{10+x}{5}:4x-9::2:5$, to find the value of x. *Ans.* $x=4$.

Ex. 18. Given $\frac{17-4x}{4}:\frac{15+2x}{3}-2x::5:4$, to find the value of x. *Ans.* $x=3$.

Ex. 19. Given $\frac{7x+5}{6}:\frac{6x-3}{7}::\frac{3+6x}{2}:\frac{5-7x}{4}$, to find the value of x. *Ans.* $x=\frac{1}{5}\sqrt{\frac{283}{31}}$.

Ex. 20. Given $\sqrt{\{4+\sqrt{x^4-x^2}\}}=x-2$, to find the value of x. *Ans.* $x=2\frac{1}{8}$.

Ex. 21. Given $\sqrt{a+\sqrt{x}}+\sqrt{a-\sqrt{x}}=\sqrt{x}$, to find the value of x. *Ans.* $x=4(a-1)$.

Ex. 22. Given $\sqrt{\left(\frac{a^2}{x}+b\right)}-\sqrt{\left(\frac{a^2}{x}-b\right)}=\sqrt{c}$, to find the value of x. *Ans.* $x=\frac{4a^2c}{4b^2+c^2}$.

Ex. 23. Given $\sqrt{a+x}-\sqrt{\frac{a}{a+x}}=\sqrt{2a+x}$, to find the value of x. *Ans.* $x=\left(\frac{1-2\sqrt{a}-a}{2+\sqrt{a}}\right)\sqrt{a}$.

119. In all cases where there are two or more unknown quantities, there must be as many independent equations, otherwise the number of results is unlimited.

If, for example, the equation

$$5x+6y=11$$

were given, to determine the values of x and y, we would have
$$x = \frac{11 - 6y}{5},$$
in which we might give to y any value whatever, and thence determine a corresponding value of x. (See *Indeterminate Analysis.*)

But, if to this we add the condition expressed in the following equation, viz.:
$$3x - 2y = 5,$$
the problem becomes entirely limited; for from this we derive
$$x = \frac{5 + 2y}{3},$$
and as these two values must be equal, we have
$$\frac{11 - 6y}{5} = \frac{5 + 2y}{3},$$
from which we readily determine the value of y, viz.:
$$y = \frac{2}{7}$$
whence $x = \dfrac{11 - 6y}{5} = \dfrac{13}{7}$, or $x = \dfrac{5 + 2y}{3} = \dfrac{13}{7}$, as before.

Moreover, these are the only values which will answer the required conditions.

120. The above solution leads to

METHOD 1,

Of determining the values of the unknown quantities in two equations.

It consists in finding the values of one of the unknown quantities in each equation, and equating the results. We thus obtain an equation which contains but one unknown quantity. This may be determined by the methods already explained.

This process, by which we arrive at an equation independent of one of the unknown quantities, is called *elimination*, and the quantity which is thus made to disappear is said to be *eliminated*.

Method 2.

Multiply one or both of the equations by such number or numbers as will make the coefficients of one of the unknown quantities the same in both; then add or subtract, according as these terms have different or like signs. By this process one of the unknown quantities will be eliminated; and the value of the other may be determined as before.

As an example of this method, which is generally the best, as it admits of more compactness, we shall solve the same equations as before, viz.:

$$5x + 6y = 11$$
$$3x - 2y = 5.$$

Multiplying the last by 3, we have

$$9x - 6y = 15,$$

adding to the first $\quad 14x = 26;$

whence $\quad x = \frac{13}{7}$

and $\quad y = \frac{11 - 5x}{6} = \frac{2}{7}$ as before.

We may also eliminate by

Method 3.

Find the value of one of the unknown quantities in either of the equations, and substitute the value thus determined in the other equation. We will thus arrive at an equation containing only the other unknown quantity, which may be found as before.

Let the equations be

$$5x + 6y = 11$$
$$3x - 2y = 5.$$

From the last equation, we have

$$x = \frac{5 + 2y}{3}.$$

This, substituted in the first equation, gives

$$\frac{25 + 10y}{3} + 6y = 11.$$

Clearing of fractions $25 + 10y + 18y = 33$,

whence $28y = 8$,

and $y = \frac{2}{7}$,

Consequently $x = \frac{5 + 2y}{3} = \frac{13}{7}$.

121. Whichever of these above methods we desire to apply, it will generally be best to clear the equation of fractions, and collect the terms which contain the unknown quantities. If method 2 is employed, they should be upon the left side of the equation.

Though the second is generally the best method, yet it is not universally so, as in some cases particular artifices may be employed which will shorten the process.

If, for example, the values of x and y were required from the equations

$$\frac{4}{x} + \frac{5}{y} = \frac{9}{y} - 1$$
$$\frac{5}{x} + \frac{4}{y} = \frac{7}{x} + \frac{3}{2},$$

and we were to clear them of fractions, we would obtain

$$4y + 5x = 9x - xy$$
$$10y + 8x = 14y + 3xy,$$

which, by transposition, become

$$4x - 4y = xy$$
$$8x - 4y = 3xy,$$

whence, by subtraction,

$$4x = 2xy,$$

or $y = 2$.

From this we readily derive

$$x = 4.$$

Otherwise. Transposing, we have $\frac{4}{x} - \frac{4}{y} = -1$,

and $\frac{2}{x} - \frac{4}{y} = -\frac{3}{2}$,

whence, by subtraction, $\frac{2}{x} = \frac{1}{2}$,

and $x = 4$,

$y = 2$, as before.

EXAMPLES.

Ex. 1. Given $\left\{\begin{matrix} 7x + 3y = 29 \\ 5x + 2y = 20 \end{matrix}\right\}$ to determine the values of x and y. *Ans.* $x = 2$, $y = 5$.

Ex. 2. Given $\left\{\begin{matrix} 4x - 7y = 34 \\ 8x + 3y = 102 \end{matrix}\right\}$ to find the values of x and y. *Ans.* $x = 12$, $y = 2$.

Ex. 3. Given $\left\{\begin{matrix} 12x + 3y = 5y + 36 \\ \frac{4x}{5} - \frac{2y - 3}{7} = 4y - \frac{8x + 15}{5} - 36 \end{matrix}\right\}$ to find the values of x and y. *Ans.* $x = 5$, $y = 12$.

Ex. 4. Given $\left\{\begin{matrix} \frac{x}{7} + 7y = 99 \\ \frac{y}{7} + 7x = 51 \end{matrix}\right\}$ to find the values of x and y. *Ans.* $x = 7$, $x = 14$.

Ex. 5. Given $\left\{\begin{matrix} \frac{1}{3}x - \frac{1}{4}y = 2 \\ \frac{1}{4}x + \frac{1}{2}y = 7 \end{matrix}\right\}$ to find the values of x and y. *Ans.* $x = 12$, $y = 8$.

Ex. 6. Given $\left\{\begin{matrix} x + 1 : y :: 5 : 3 \\ \frac{2x}{3} - \frac{5 - y}{2} = \frac{41}{12} - \frac{2x - 1}{4} \end{matrix}\right\}$ to find the values of x and y. *Ans.* $x = 4$, $y = 3$.

Ex. 7. Given $\left\{\begin{matrix} \frac{7 + x}{5} - \frac{2x - y}{4} = 3y - 5 \\ \frac{5y - 7}{2} + \frac{4x - 3}{6} = 18 - 5x \end{matrix}\right\}$ to find the values of x and y. *Ans.* $x = 3$, $y = 2$.

Ex. 8. Given $\left\{\begin{matrix} x + y : 4x + y :: 4 : 7 \\ \frac{\frac{11y}{6} - 2x}{5} + \frac{21 - 3y}{4} = \frac{2 + \frac{x}{10}}{3} - \frac{1}{6} \end{matrix}\right\}$ to find the values of x and y. *Ans.* $x = 3$, $y = 9$.

Ex. 9. Given $\left\{\begin{matrix} ax + by = m \\ a'x + b'y = m' \end{matrix}\right\}$ to find the values of x and y. *Ans.* $x = \frac{b'm - bm'}{ab' - a'b}, y = \frac{am' - a'm}{ab' - a'b}$.

These results will serve to solve all equations that can be reduced to forms similar to those of this example; we have merely to substitute for a, a', b, b', m and m', the numbers to which they are equal, and then reduce.

Ex. 10. Given $\left\{\begin{matrix} \frac{a}{x} + \frac{b}{y} = m \\ \frac{c}{x} + \frac{d}{y} = n \end{matrix}\right\}$ to find the values of x and y.* *Ans.* $x = \frac{bc - ad}{nb - md}, y = \frac{bc - ad}{mc - na}$.

Ex. 11. Given

$$\left\{\begin{matrix} y + 2x : y - 2x :: 12x + 6y - 3 : 6y - 12x - 1 \\ \frac{5x + 3y}{4} - \frac{3x + \frac{y}{7}}{5} = \frac{12x - 3y}{7} + 3 + \frac{15x}{28} \end{matrix}\right\}$$

to find the values of x and y. *Ans.* $x = 1, y = 4$.

Ex. 12. Given $\left\{\begin{matrix} \frac{a}{b + y} = \frac{b}{3a + x} \\ ax + 2by = c \end{matrix}\right\}$ to find the values of x and y. *Ans.* $x = \frac{2b^2 - 6a^2 + c}{3a}$.

$$y = \frac{3a^2 - b^2 + c}{3b}.$$

Ex. 13. Given

$$\left\{\begin{matrix} 3x + 5a : 3x - 5a :: 4x + 2a - y : 4x - 2a + y \\ 7ax - 3y : 7ay - 5x :: 7ax + 3y : 7ay + 5x \end{matrix}\right\}$$

to find the values of x and y.

Ans. $x = -\frac{14a}{15}\sqrt{15}$.

$$y = -\frac{14a}{3}.$$

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* See example wrought, page 107.

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Ex. 14. Given $\begin{cases} x + y = a \\ x^2 - y^2 = b \end{cases}$ to find x and y.

Ans. $x = \frac{a^2 + b}{2a}$, $y = \frac{a^2 - b}{2a}$.

Ex. 15.

Given $\left\{\begin{array}{l} \frac{\frac{7x}{4} + 6y}{5} - \frac{\frac{3y+6}{5} - \frac{3x-2}{10}}{8} = 5 - \frac{x}{16} \\ \frac{3x}{2} + \frac{2y}{3} + \frac{5}{2} : \frac{x}{2} - \frac{y}{3} + \frac{1}{6} :: 10\frac{1}{2} : 1\frac{1}{6} \end{array}\right\}$

to find the values of x and y.

Ans. $x = 4$.
$y = 3$.

Ex. 16. Given $\left\{\begin{array}{l} 2\sqrt{x} + \frac{1}{2}y - 13 = \sqrt{x} - \frac{1}{2}y + 9 \\ 9\sqrt{x} + \frac{3}{2}y - 10 = \frac{1}{2}\sqrt{x} + \frac{1}{2}y + 72 \end{array}\right\}$

to find the values of x and y.

Ans. $x = 64$, $y = 14$.

Ex. 17. Given $\left\{\begin{array}{l} 4x - 34\frac{1}{3} - \frac{4y + 13x}{27 - 6y} = \frac{12x + 8}{3} \\ 3x + \frac{21 - 4y}{4x - 10} = \frac{18x + 13}{6} - 2\frac{1}{9} \end{array}\right\}$

to find the values of x and y. *Ans.* $x = 7$, $y = 5$.

Ex. 18. Given

$\left\{\begin{array}{l} \frac{3x+2y}{5} - \frac{5x - \frac{3y}{4} + 1}{3} = x + \frac{y - 2x}{10} - \frac{4x - y}{7} \\ y + 2x : y - 2x :: 12x + 6y - 3 : 6y - 12x - 1 \end{array}\right\}$

to find the values of x and y. *Ans.* $x = 1$, $y = 4$.

Ex. 19. Given $x - \frac{2y - x}{23 - x} = 20 - \frac{59 - 2x}{2}$,

and $y + \frac{y - 3}{x - 18} = 30 - \frac{73 - 3y}{3}$, to find

the values of x and y. *Ans.* $x = 21$, $y = 20$.

Ex. 20. Given $\begin{cases} 4x + \dfrac{15-x}{4} = 2y + 5 + \dfrac{7x+11}{16} \\ 3y - \dfrac{2x+y}{5} = 2x + \dfrac{2y+4}{3} \end{cases}$

to find the values of x and y.

Ans. $x = 3, y = 4$.

Ex. 21. Given $\begin{cases} 3x + 6y + 1 = \dfrac{6x^2 + 130 - 24y^2}{2x - 4y + 3} \\ 3x - \dfrac{151 - 16x}{4y - 1} = \dfrac{9xy - 110}{3y - 4} \end{cases}$

to find the values of x and y.

Ans. $x = 9, y = 2$.

122. *Equations containing three or more unknown quantities.*

The methods employed in solving equations containing three or more unknown quantities, being only an extension of those explained in the last article, require no farther elucidation. We shall therefore merely apply them to a few examples, and affix others for the exercise of the student.

EXAMPLES.

Ex. 1. Given $\begin{cases} x + y + z = 15 \\ x + 2y + 3z = 23 \\ x + 3y + 4z = 28 \end{cases}$ to find the values of x, y, and z.

Subtracting the 1st from the 2d, and the 2d from the 3d, we have

$$y + 2z = 8$$
$$y + z = 5,$$

whence

$$z = 3$$
$$y = 2$$

and

$$x = 15 - y - z = 10.$$

Ex. 2. Given

$$2x + 3y + 4z = 40$$
$$4x + 5y + 6z = 90$$
$$5x - 3y + 4z = 75,$$

to find the values of x, y, and z.

Subtracting the 1st from the third, we have

$$3x - 6y = 35.$$

Multiply the 1st by 3, and the 2d by 2, and subtract the former result from the latter, and

$$2x + y = 60.$$

Whence $\qquad 12x + 6y = 360,$

but $\qquad 3x - 6y = 35,$

$\therefore \qquad 15x = 395,$

and $\qquad x = 26\frac{1}{3},$

Also $\qquad y = \dfrac{3x - 35}{6} = 7\frac{1}{3},$

and $\qquad z = \dfrac{40 - 3y - 2x}{4} = -8\frac{2}{3}.$

Ex. 3. Given $\quad x + \frac{1}{2}y + \frac{1}{3}z = 32$

$$\frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z = 15$$

$\frac{1}{4}x + \frac{1}{5}y + \frac{1}{6}z = 12$, to find the values of x, y, and z.

Ans. $x = 12$, $y = 20$, and $z = 30$.

Ex. 4. Given $\quad 7x + 5y + 2z = 79$

$$8x + 7y + 9z = 122$$

$x + 4y + 5z = 55$, to find the values of x, y, and z.

Ans. $x = 4$, $y = 9$, and $z = 3$.

This example is most conveniently solved by adding the equations together, and dividing the result by 16; also, taking the difference between the 1st and 2d, the two results and the third will form three equations, in each of which the coefficient of x will be 1.

Ex. 5. Given $\quad 4x - 3y + 2z = 10$

$$5x + 6y - 8z = -1$$

$-x + 8y + 3z = 44$, to find the values of x, y, and z.

Ans. $x = 3$, $y = 4$, and $z = 5$.

Ex. 6. Given $x+y+z=a$
$x+y+u=b$
$x+z+u=c$
$y+z+u=d$, to find the values of x, y, and z.

Ans. $x=\frac{a+b+c-2d}{3}$
$y=\frac{a+b+d-2c}{3}$
$z=\frac{a+c+d-2b}{3}$
$u=\frac{b+c+d-2a}{3}$.

Ex. 7. Given

$$\left.\begin{array}{l}\frac{4x+3y+3}{10}-\frac{2x+2z-y+1}{15}=5-\frac{x-z-5}{5}\\ \frac{9x+5y-2z}{12}-\frac{2x+y-3z}{4}=\frac{7y+z+3}{11}+\frac{1}{6}\\ \frac{5y+3z}{4}-\frac{2x+3y-z}{12}+2z=y-1+\frac{3x+2y+7}{6},\end{array}\right\}$$

to find the values of x, y, and z.

Ans. $x=9$, $y=7$, and $z=3$.

Ex. 8. Given $\left\{\begin{array}{l}\frac{1}{x}+\frac{1}{y}=\frac{1}{8}\\ \frac{1}{x}+\frac{1}{z}=\frac{1}{9}\\ \frac{1}{y}+\frac{1}{z}=\frac{1}{10}\end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=14\frac{34}{49}$, $y=17\frac{23}{41}$, and $z=23\frac{7}{31}$.

Ex. 9. Given $\left\{\begin{array}{l}xyz=24\\ vxy=30\\ vxz=40\\ vyz=60\end{array}\right\}$ to find the values of v, x, y, and z.

Ans. $v=5$, $x=2$, $y=3$, and $z=4$.

Ex. 10. Given $\left\{\begin{array}{l} x(y+z)=96 \\ y(x+z)=105 \\ z(x+y)=117 \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=6$, $y=7$, and $z=9$.

123. The following method is sometimes preferable, especially where the coefficients are literal.

Multiply the 2d, 3d, &c., equations by some indeterminate quantities, and add the results to the 1st. Assume the coefficients of all the unknown quantities except one in the sum, equal to 0, we will thus be furnished with a number of new equations equal to the indeterminate quantities introduced, from which equations these may be found. Their values substituted into the sum of the equations will eliminate all the unknown quantities except one, whence this may be determined.

EXAMPLES.

Ex. 1. Given

$$2x+5y+7z=45$$
$$3x+2y-3z=-3$$
$$5x+3y-z=10,$$

to find the values of x, y, and z.

Multiply the 2d by m, and the third by n, and we have

$$2x+5y+7z=45$$
$$3mx+2my-3mz=-3m$$
$$5nx+3ny-nz=10n.$$

Assume now $2m+3n+5=0$,

and $-3m-n+7=0$,

or $2m+3n=-5$,

and $3m+n=7$,

whence $m=3\frac{5}{7}$ and $n=-4\frac{1}{7}$;

consequently $2x+3mx+5nx=45-3m+10n$

becomes $2x+\frac{78}{7}x-\frac{145}{7}x=45-\frac{78}{7}-\frac{290}{7}$,

or $14x+78x-145x=315-78-290$,

whence $53x=53$,

and $x=1$.

In like manner $y=3$

$$z=4.$$

Ex. 2. Given
$$ax+by=c$$
$$a'x+b'y=c'.$$

Multiplying the latter by m and adding to the former, we have

$$(a+ma')x+(b+mb')y=c+mc'.$$

If now we assume $b+mb'=0$, or $m=-\frac{b}{b'}$, we have

$$x=\frac{c+mc'}{a+ma'}=\frac{b'c-bc'}{ab'-a'b},$$

and
$$y=\frac{ac'-a'c}{ab'-a'b}.$$

Ex. 3. Given $\left\{\begin{array}{l} a'x+b'y+c'z=m' \\ a''x+b''y+c''z=m'' \\ a'''x+b'''y+c'''z=m''' \end{array}\right\}$ to find the values of x, y, and z.

Multiplying the second by p, and the third by q, the equations become

$$\left.\begin{array}{l} a'x+b'y+c'z=m' \\ pa''x+pb''y+pc''z=pm'' \\ qa'''x+qb'''y+qc'''z=qm'''. \end{array}\right\} \quad \text{(A)}$$

Assume now $pb''+qb'''=-b'$
and $pc''+qc'''=-c'$, (B)

and we will have for the sum of the equations (A),

$$a'x+pa''x+qa'''x=m'+pm''+qm''',$$

whence
$$x=\frac{m'+pm''+qm'''}{a'+pa''+qa'''}, \quad \text{(C)}$$

in which the only difference between the numerator and denominator is, that the former contains the absolute quantities m', m'', and m''', in place of the coefficients of x, viz., a', a'', and a'''.

From equations (B) we readily find, as in last example,

$$p=\frac{b'''c'-b'c'''}{b''c'''-b'''c''},$$

and

$$q=\frac{b'c''-b''c'}{b''c'''-b'''c''},$$

which, substituted in (C), give

$$x=\frac{m'+m''\dfrac{b'''c'-b'c'''}{b''c'''-b'''c''}+m'''\dfrac{b'c''-b''c'}{b''c'''-b'''c''}}{a'+a''\dfrac{b'''c'-b'c'''}{b''c'''-b'''c''}+a'''\dfrac{b'c''-b''c'}{b''c'''-b'''c''}}$$

$$=\frac{m'b''c'''-m'b'''c''+m''b'''c'-m''b'c'''+m'''b'c''-m'''b''c'}{a'b''c'''-a'b'''c''+a''b'''c'-a''b'c'''+a'''b'c''-a'''b''c'}.$$

In a similar manner we may find

$$y=\frac{m'a'''c''-m'a''c'''+m''a'c'''-m''a'''c'+m'''a''c'-m'''a'c''}{b'a'''c''-b'a''c'''+b''a'c'''-b''a'''c'+b'''a''c'-b'''a'c''},$$

and

$$z=\frac{m'a''b'''-m'a'''b''+m''a'''b'-m''a'b'''+m'''a'b''-m'''a''b'}{c'a''b'''-c'a'''b''+c''a'''b'-c''a'b'''+c'''a'b''-c'''a''b'}.$$

If we examine the denominators of these values we will readily see that they are identical, and consist of all the combinations of the coefficients of x, y, and z, that can be made by taking a coefficient of x, one of y, and one of z, from the successive equations; the signs being alternately *plus* and *minus*.

The numerators are evidently the same as the denominators, except, that instead of the coefficients of that unknown quantity to which the result belongs, the absolute number in the same equation is substituted.

Now, if we arrange the coefficients as in the following table, and take one from each column, proceeding diagonally downwards and to the right, and then repeat the process diagonally upwards, we shall form all the combinations of the kind desired for the denominator.

The denominator will be equal to the sum of the combinations taken downwards, diminished by the sum of those proceeding upwards. Thus,

$$\begin{array}{lll} a' & b' & c' \\ a'' & b'' & c'' \\ a''' & b''' & c''' \\ a' & b' & c' \\ a'' & b'' & c''. \end{array}$$

The combinations downwards and to the right, are

$$a'b''c''', \ a''b'''c', \ a'''b'c''.$$

Those taken upwards and to the right, are

$$a''b'c''', \ a'b'''c'', \ a'''b''c'.$$

And the denominator is

$$a'b''c'''+a''b'''c'+a'''b'c''-a''b'c'''-a'b'''c''-a'''b''c',$$

as will be seen by comparing with the values before obtained.

To obtain the numerators—that of y for instance—write the numbers m', m'', and m''', instead of b', b'', and b''', in the table, and take the products in the same manner. Thus,

$$\begin{array}{lll} a' & m' & c' \\ a'' & m'' & c'' \\ a''' & m''' & c''' \\ a' & m' & c' \\ a'' & m'' & c''. \end{array}$$

The numerator will be

$$a'm''c'''+a''m'''c'+a'''m'c''-a''m'c'''-a'm'''c''-a'''m''c'.$$

The above process will be found very much to shorten the process of solution, in equations containing three unknown quantities.

EXAMPLES.

Ex. 1. Given $\begin{cases} 5x+3y+2z=29 \\ 2x+5y-z=14 \\ 3x-2y+4z=20 \end{cases}$ to find the values of x, y, and z.

First, for the denominator,

$$\begin{array}{rrr} 5, & 3, & 2, \\ 2, & 5, & -1, \\ 3, & -2, & 4, \\ 5, & 3, & 2, \\ 2, & 5, & -1. \end{array}$$

Positive products, $5 \times 5 \times 4 + 2 \times -2 \times 2 + 3 \times 3 \times -1$
$= 100 - 8 - 9 = 83.$

Negative " $2 \times 3 \times 4 + 5 \times -2 \times -1 + 3 \times 5 \times 2$
$= 24 + 10 + 30 = 64,$

and the denominator is

$$83 - 64 = 19.$$

2d. For numerator of x,

$$\begin{array}{rrr} 29, & 3, & 2, \\ 14, & 5, & -1, \\ 20, & -2, & 4, \\ 29, & 3, & 2, \\ 14, & 5, & -1. \end{array}$$

Positive products, $29 \times 5 \times 4 + 14 \times -2 \times 2 + 20 \times 3 \times -1$
$= 580 - 56 - 60 = 464.$

Negative " $14 \times 3 \times 4 + 29 \times -2 \times -1 + 20 \times 5 + 2$
$= 168 + 58 + 200 = 426.$

The numerator, therefore, is $464 - 426 = 38$.

Consequently $x = \frac{38}{19} = 2.$

Similarly the numerator for y is
$(5 \times 14 \times 4 + 2 \times 20 \times 2 + 3 \times 29 \times -1)$
$-(2 \times 29 \times 4 + 5 \times 20 \times -1 + 3 \times 14 \times 2) = 273 - 216 = 57,$

and $y = \frac{57}{19} = 3.$

Also, for z, the numerator is
$(5 \times 5 \times 20 + 2 \times -2 \times 29 + 3 \times 3 \times 14)$
$-(2 \times 3 \times 20 + 5 \times -2 \times 14 + 3 \times 5 \times 29) = 510 - 415 = 95,$

and $$z = \frac{95}{19} = 5.$$

Ex. 2. Given $\left\{\begin{array}{l} 3x - 6y + z = -32 \\ 2x + 8y = 16 \\ 5x + 7y - 2z = 5 \end{array}\right\}$ to find the values of x, y, and z.

First, The denominator

$$= (3\times8\times-2+2\times7\times1+5\times-6\times0)$$
$$-(5\times8\times1+2\times-6\times-2+3\times7\times0)$$
$$= -34 - 64 = -98.$$

The numerator of $x =$

$$(-32\times8\times-2+16\times7\times1+5\times-6\times0)$$
$$-(5\times8\times1+16\times-6\times-2+-32\times7\times0)$$
$$= 624 - 232 = 392.$$

The numerator of $y =$

$$(3\times16\times-2+2\times5\times1+5\times-32\times0)$$
$$-(5\times16\times1+2\times-32\times-2+3\times5\times0)$$
$$= -86 - 208 = -294.$$

The numerator of $z =$

$$(3\times8\times5+2\times7\times-32+5\times-6\times16)$$
$$-(5\times8\times-32+2\times-6\times5+3\times7\times16)$$
$$= -808 + 1004 = 196.$$

Therefore,

$$x = \frac{392}{-98} = -4,\ y = \frac{-294}{-98} = 3 \text{ and } z = \frac{196}{-98} = -2.$$

Ex. 3. Given $\left\{\begin{array}{l} 3x - 9y + 8z = 41 \\ -5x + 4y + 2z = -20 \\ 11x - 7y - 6z = 37 \end{array}\right\}$ to find the value of x, y, and z.

Ans. $x = 2$, $y = -3$, and $z = 1$.

Ex. 4. Given $\left\{\begin{array}{l}\frac{x+y}{3}+2z=21\\ \frac{y+z}{2}-3x=-65\\ \frac{3x+y-z}{2}=38\end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=24$, $y=9$, and $z=5$.

Ex. 5. Given $\left\{\begin{array}{l}\frac{2}{3}x-\frac{1}{2}y+\frac{3}{4}z=4\\ \frac{1}{2}x+\frac{2}{3}y-3z=-13\\ 3x-2y+z=2\end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=6$, $y=12$, and $z=8$.

SECTION II.

Questions producing Simple Equations.

In solving such questions, it is impossible to give any rules that will guide in all, or even in a majority of cases. Almost every thing depends upon the ingenuity of the student, upon his habits of analysis, and a judicious selection of the quantity to be represented by the unknown.

We should, after deliberately studying the question, fix upon something as the quantity to be determined, and then operate upon the letter by which it is represented, as though it were known, and we wished to prove the result. We will thus arrive at an equation which, when solved, will determine the value of the required quantity.

Example 1.

To find two numbers in the ratio of 3 to 5; such that their sum shall be equal to one-fourth the difference of their squares.

This may be solved in various ways,

1st. Let x represent the smaller number
and y the larger;

then we have $x : y :: 3 : 5$, or $y = \frac{5}{3}x$

and $\frac{1}{4}y^2 - \frac{1}{4}x^2 = x + y$,

dividing by $\frac{1}{4}y + \frac{1}{4}x$, we have

$$y - x = 4:$$

or, substituting the value of y, above determined,

$$\frac{5}{3}x - x = \frac{2}{3}x = 4,$$

whence $x = 6$

and $y = \frac{5}{3}x = 10$;

or, more simply, thus,

Let $5x$ and $3x$ represent the numbers,

then we have $5x + 3x = \frac{1}{4}(25x^2 - 9x^2)$,

or, $8x = 4x^2$,
whence $x = 2$
and $5x = 10$
$3x = 6$
as before.

The above solutions lead to the important remark, that in selecting the unknowns we should endeavour to embody as many of the conditions as possible in the numbers assumed.

The following example affords a further illustration of this rule.

Ex. 2. To divide the number 116 into four parts, such that if the first be increased by 5, the second diminished by 4, the third multiplied by 3, and the fourth divided by 2, the results shall be equal.

Let $x-5$, $x+4$, $\frac{x}{3}$ and $2x$ represent the parts. It is evident that all the conditions, except the first, viz., that the parts make 116, are fulfilled by these numbers, whatever value x may have; the result in each case being x. It only remains to fulfil the other condition, which leads to the equation

$$x-5+x+4+\frac{x}{3}+2x=116;$$

or $13x=351$,

whence $x=27$

and the parts are 22, 31, 9 and 54.

Ex. 3. Divide the number 49 into two such parts, that the greater, increased by 6, may be to the less diminished by 11 as 9 to 2.

Let $9x-6=$ the greater

and $2x+11=$ the less.

These fulfil the last condition, since the first $+6=9x$ and the second $-11=2x$, which results are as 9 to 2.

The first condition gives

$$9x-6+2x+11 \text{ or } 11x+5=49,$$

whence $x=4$

$9x-6=30$

and $2x+11=19$.

Another Solution.

Let $x=$ the greater, and $49-x=$ the less, then we have

$$x+6:38-x::9:2,$$

whence (Art. 47) $x+6:44::9:11$

and (Art. 45) $x+6:4::9:1$,

whence $x+6=36$

and $x=30$

$\therefore$ $49-x=19$,

Ex. 5. A courier having been gone 6 days, travels 60 miles per day. A second is then sent to overtake him, and travels 80 miles per day. In what time will he overtake the former, and how far will they have travelled?

Let $x =$ the number of days the second travels;
then $x + 6 =$ the number the first travels;
$\therefore$ $80\,x =$ the number of miles the second travels;
and $60\,(x + 6) = 60\,x + 360 =$ the number the first travels.
Hence $80\,x = 60\,x + 360$
and $20\,x = 360$,
or $x = 18$, the number of days,
$80\,x = 1440$, the distance.

Ex. 6. Says A to B, give me 10 dollars of your money, and I shall have three times as much as you. What was the amount each had, supposing the whole amount is \$110.
Let $x =$ the number of dollars A had,
then $110 - x =$ the number B had,
Then, after B had given 10 to A, they would have $x + 10$ and $100 - x$, respectively.
$\therefore$ by the question

$$x + 10 = 3\,(100 - x) = 300 - 3\,x,$$

or $4\,x = 290$
and $x = 72\frac{1}{2}$ the number of dollars A had,
$110 - x = 37\frac{1}{2}$ the number B had.

Ex. 7. A gentleman whose property is all invested in securities bearing 5 per cent. interest, spends $\frac{3}{5}$ of his income in his household expenses; $\frac{1}{3}$ of it in clothing. The balance, which is \$820, is laid out for charitable purposes. What is the value of his estate.

Let x be the value of his estate in dollars,

then $\frac{5}{100}x = \frac{x}{20}$ is his yearly income,

$\therefore$ $\frac{3}{5}$ of $\frac{x}{20} = \frac{3\,x}{100} =$ the sum spent for household purposes,

$\frac{1}{3}$ of $\frac{x}{20} = \frac{x}{60} =$ the sum laid out in clothing.

Consequently $\frac{x}{20} = \frac{3\,x}{100} + \frac{x}{60} + 820.$

Multiplying by 300, to clear of fractions, we have

$$15\,x = 9\,x + 5\,x + 246000,$$

whence $x = 246000$, the value of the estate.

Ex. 8. A can perform a piece of work in 20 days, B and C can together do it in 12 days. Now, if they all work for 6 days, C can finish it in 3 days. In what time would B or C have done it alone?

Let $x =$ the number of days which B would require,

and $y =$ the number C would require,

then $\frac{1}{x}$ and $\frac{1}{y}$ are the parts of the work which each will perform in 1 day.

The three will therefore, in the 6 days, do

$$\frac{6}{20}+\frac{6}{x}+\frac{6}{y}.$$

Consequently, $\frac{6}{20}+\frac{6}{x}+\frac{6}{y}+\frac{3}{y}=\frac{6}{20}+\frac{6}{x}+\frac{9}{y}$ will represent the whole work.

also, $\frac{12}{x}+\frac{12}{y}$ will represent the whole work, since A and B can do it in 12 days. We have then the equations

$$\frac{12}{x}+\frac{12}{y}=1$$

and $$\frac{6}{20}+\frac{6}{x}+\frac{9}{y}=1.$$

Multiplying the last by 2, and transposing, it becomes

$$\frac{12}{x}+\frac{18}{y}=\frac{7}{5},$$

whence, by subtraction, $\frac{6}{y}=\frac{2}{5}$

and $y = 15$, the number of days C requires;

also, $x = 60$, the number B requires.

2d Solution. Since B and C can do the whole work in 12 days, it is evident they will do the half of it in 6 days; also, A in 6 days does $\frac{6}{20}=\frac{3}{10}$ of the work; hence the three will perform $\frac{1}{2}+\frac{3}{10}=\frac{4}{5}$ of it in the 6 days;

and since C in three days does the $\frac{3}{y}$th part we will have

$$\frac{3}{y}+\frac{4}{5}=1,$$

whence $$\frac{3}{y}=\frac{1}{5}$$

and $$y=15, \text{ as before;}$$

also, since $\frac{12}{x}+\frac{12}{y}=1$ we have, by substituting the value of y, and clearing of fractions,

$$x=60.$$

Ex. 9. The specific gravity of tin being 7·4, and that of lead 11·5. Required the number of pounds of each metal in a mass of solder weighing 120 lbs., of the specific gravity $8\frac{1}{4}$.

NOTE. The specific gravity of a body is its weight compared with that of an equal bulk of distilled water, at the temperature of 40° of Fahrenheit.

The specific gravity is obtained by determining the weight of the body in the air, and then weighing it in water, the loss being the weight of an equal bulk of water. Having thus found the weight of the body, and also that of the same volume of water, the former of these is divided by the latter for the specific gravity.

Thus, suppose a mass of a certain mineral weighs 14 oz. in the air, and 12 oz. in water; the loss, 2 oz., is the weight of an equal bulk of water. Hence $\frac{14}{2}=7$ is the specific gravity.

In the combinations of different metals, the specific gravity of the compound is not generally the mean between those of its constituents, as the present question supposes, but it is rather greater in most cases. The amount of condensation in each instance can be determined only by experiment.

Let x be the number of pounds of tin,
and y of lead.

Then the weight of the water displaced by x pounds of tin and y pounds of lead, will be respectively

$$\frac{x}{7\cdot4} \text{ and } \frac{y}{11\cdot5}.$$

Also, 120 lbs. of the solder, of the specific gravity $8\frac{4}{7}$, will displace $120 \div 8\frac{4}{7} = 14$ lbs. of water.

We therefore have the equations

$$x + y = 120$$

$$\frac{x}{7\cdot4} + \frac{y}{11\cdot5} = 14.$$

Clearing the last of fractions, it becomes

$$115\,x + 74\,y = 11914,$$

but $\quad 74\,x + 74\,y = 8880,$

whence $\quad 41\,x = 3034,$

and $\quad x = 74 =$ number of pounds of tin,

$\therefore \quad y = 120 - x = 46,$ number of pounds of lead.

Ex. 10. A person has $10000 invested in the stock of two banks, A and B. The first year the institution A, in which the smaller sum is invested, divided 2 per cent. more than the other. The second year the dividends made by A were decreased, and those made by B were increased 1 per cent.; the receipts from both together being thus increased by $\frac{1}{4}$th of their former value. The third year B divided the same as the second year, and A the same as it did the first, the whole being by this means increased by $\frac{1}{3}$d of its original value. What was the sum invested in each bank, and the rate per cent.?

Let, $\quad x =$ the less sum,

$\therefore \quad 10000 - x =$ the greater.

Let $\quad y + 1 =$ the rate of dividend on the less,

$y - 1 =$ the rate on the greater.

Then $\dfrac{x(y+1)}{100} + \dfrac{(10000 - x)(y - 1)}{100} = \dfrac{x}{50} + 100\,y$ $- 100 =$ the whole amount of dividend the first year, and $\frac{xy}{100} + \frac{(10000-x)y}{100} = 100y =$ the amount of dividend the second year.

$$\therefore \quad \frac{x}{50} + 100y - 100 + \frac{1}{4}\left(\frac{x}{50} + 100y - 100\right) = 100y,$$

or $$\frac{x}{50} - 100 + \frac{x}{200} + 25y - 25 = 0,$$

whence $$\frac{x}{40} + 25y = 125,$$

and $$x + 1000y = 5000. \qquad \text{(A)}$$

Again, the whole dividend the third year is

$$\frac{x(y+1)}{100} + \frac{(10000-x)y}{100} = \frac{x}{100} + 100y.$$

And this being equal to the 1st year's dividend, increased by ⅓d, we have

$$\frac{x}{50} + 100y - 100 + \frac{1}{3}\left(\frac{x}{50} + 100y - 100\right) = \frac{x}{100} + 100y,$$

or $$\frac{x}{20} + 100y = 400.$$

Subtracting 10 times this from equation (A), we obtain

$$-\frac{1}{2}x = 1000,$$

whence $x = 2000 =$ the sum in the bank A,

and $10000 - x = 8000 =$ the sum in the bank B.

Substituting the value of x thus obtained in equation A, we obtain

$$y = 3,$$

whence $\left.\begin{array}{l} y+1 = 4 \\ y-1 = 2 \end{array}\right\}$ the rates of interest.

Ex. 11. A miller had some wheat, which cost him $1.10 per bushel, and some oats, which cost 75 cents per bushel. He ground them together, and made 19 barrels of flour, which he sells for $5.997 per barrel, thus clearing 14 per cent. on the cost; the value of the bran being considered equal to the cost of grinding, packing, &c. Now, supposing 5 bushels of grain make a barrel of flour, how many bushels of each kind did he grind?

Let $x =$ the number of bushels of wheat,
and $y =$ the number of rye,
then $x + y = 19 \times 5 = 95$. (A)
Also $110\,x =$ the cost of the wheat,
and $75\,y =$ the cost of the rye.

$$\therefore\quad 110\,x + 75\,y + \frac{14}{100}(110\,x + 75\,y)$$

$$= \frac{1254}{10}x + \frac{171}{2}y = \text{the price for which the}$$

flour must be sold.

$$\therefore\quad \frac{1254}{10}x + \frac{171}{2}y = 599.7 \times 19 = 11394.3,$$

or $1254\,x + 855\,y = 113943$.
But (A) $855\,x + 855\,y = 81225$,
hence $399\,x = 32718$,
and $x = 82 =$ the number of bushels of wheat,
$\therefore\; y = 95 - x = 13 =$ the number of rye.

Ex. 12. Required to find four numbers in arithmetical progression, which, being increased respectively by 2, 4, 8, and 15, shall be in geometrical progression?

Let $x - 2$, $xy - 4$, $xy^2 - 8$, and $xy^3 - 15$, be the numbers. It is evident these fulfil the last condition, since increasing them by 2, 4, 8, and 15, makes them

$$x,\ xy,\ xy^2, \text{ and } xy^3,$$

which are in geometrical progression. It only remains, then, to determine x and y, so that the numbers shall be in arithmetical progression.

We shall, therefore, have (Art. 58),

$$xy^2 + x - 10 = 2\,xy - 8$$

$$xy^3 + xy - 19 = 2\,xy^2 - 16.$$

Multiplying the first by y, it becomes

$$xy^3 + xy - 10\,y = 2\,xy^2 - 8\,y,$$

whence $10\,y - 19 = 8\,y - 16$,
or $2\,y = 3$,
and $y = \frac{3}{2}$.

Substituting this in one of the original equations, we obtain

$$x = 8.$$

Whence the numbers are readily found to be

6, 8, 10, and 12.

Ex. 13. Divide the number 25 into two such parts that one may exceed the other by 5. What are the parts?

Ans. 10 and 15.

Ex. 14. The sum of $5500 is to be divided between A and B, in such proportion that A will receive an eagle as often as B does a dollar. What will each man's share be?

Ans. A's = $5000. B's = $500.

Ex. 15. The number 75 is divided into two parts, such that three times the greater exceeds seven times the less by 15. What are the parts?

Ans. 21 and 54.

Ex. 16. 1200 dollars is to be divided between A and B, so that A's share shall be to B's as 2 is to 7. How much should each receive?

Ans. A, 266\frac{2}{3}$, and B 933\frac{1}{3}$.

Ex. 17. A vintner wishes to fill a cask containing 125 gallons with wine, on which he will be able to clear 15 per cent. by selling it at 92 cts. per gallon. He has two parcels which cost 70 cts. and $1.50 per gallon respectively; how much of each kind must he take?

Ans. 15$\frac{5}{8}$ galls. at $1.50, and 109$\frac{3}{8}$ galls. at 70 cts.

Ex. 18. Required to find four numbers, such that $\frac{1}{4}$ the 1st, $\frac{1}{3}$ the 2d, twice the 3d, and $\frac{1}{2}$ the 4th, may be 21$\frac{1}{2}$; twice the 1st, $\frac{1}{2}$ the 2d, $\frac{1}{7}$ the 3d, and $\frac{1}{3}$ the 4th, may be 15; the first, twice the 2d, three times the 3d, and $\frac{1}{3}$ the 4th, may be 40; and the sum of the numbers may be 26.

Ans. 4, 6, 7, and 9.

Ex. 19. A and B desiring to purchase a house jointly, have just sufficient money in bank to pay for it. A says to B, If I had twice as much as I have, and $\frac{1}{3}$ of yours, I could purchase it alone. B says, If I had half yours, and \$1000 in addition to what I have, I could pay for it. What had each?

Ans. A \$2000, and B \$3000.

Ex. 20. Purchased 25 lbs. of sugar, and 36 of coffee, for \$8.04, but the price of each having fallen 1 cent per pound, I afterwards bought 2 lbs. more of the first, and 3 lbs. more of the second, for the same money. What was the price of each?

Ans. { Sugar 12 cts.
Coffee 14 cts.

Ex. 21. A and B agree to reap a field in 12 days, A being able to work only $\frac{3}{4}$ as fast as B. Finding they would be unable to finish it, at the end of 6 days they call in C, by whose aid it is performed in the stipulated time. Now had C wrought from the beginning, they would have reaped it in 9 days. In what time would each have done it alone?

Ans. A in $31\frac{1}{2}$ days, B in 42 days, and C in 18 days.

Ex. 22. A can perform a piece of work in a days, B in b days, C in c days, and D in d days. In how many days will they finish it working together?

Ans. $\frac{abcd}{abc + abd + acd + bcd}$ days.

Ex. 23. A and B engage to finish a piece of work in 15 days, but after 7 days, finding they would be unable to accomplish it, they call in C, by whose aid it is completed in 14 days. Had C worked with them from the beginning, the three would have done it in $10\frac{1}{2}$ days. In what time would C alone have done it.

Ans. 21 days.

Ex. 24. Bought linen at 60 cts. per yard, and muslin at 15 cts. per yard, amounting in all to \$11.40. I afterwards sold $\frac{1}{3}$ of the linen and $\frac{1}{4}$ of the muslin for \$3.89, having cleared 29 cents by the bargain. How many yards of each did I purchase?

Ans. 15 linen, 16 muslin.

Ex. 25. Bought 10 cows and 15 sheep for \$215. I afterwards purchased 5 cows and 7 sheep for \$107.50, the cows costing \$1 a head more, and the sheep 50 cts. a head less than before. What were the prices of the first lot?

Ans. Sheep \$3, cows \$17.

Ex. 26. There are two numbers in the ratio of 5 : 4, but if each be increased by 20, the results are as 9 : 8. What are the numbers? *Ans.* 25 and 20.

Ex. 27. What fraction is that to the numerator of which, if 1 be added, the fraction will equal $\frac{1}{3}$, but if 4 be added to the denominator, the fraction becomes $\frac{1}{4}$? *Ans.* $\frac{7}{24}$.

Ex. 28. A and B can do a piece of work in 15 days. When it was $\frac{3}{5}$ done, they called in C, with whose aid the work was finished in 12 days. In what time could C alone have done it? *Ans.* 15 days.

Ex. 29. A can do a piece of work in 20 days, and B and C can together perform it in 12 days. Now, if all three work for 6 days, C can finish it in 3 days. In what time would B or C have performed it?

Ans. B 60 days, C 15 days.

Ex. 30. A, B and C can perform a piece of work in 12 days. But after A and B had worked together 7 days, C finished it in 21 days. In what time would each have done it separately, supposing A can do as much in 4 days as B can in 5?

Ans. A in $33\frac{3}{5}$, B in 42, and C in $33\frac{3}{5}$.

Ex. 31. Two men, A and B, agree to finish a piece of work in 12 days. But after they have worked together 6 days, finding they will be unable to accomplish it, they call in C, and the three finish the work in 12 days. Now, if C had worked with A from the beginning, the two would have accomplished the work in 14 days, and B and C would have done it in 11 days. In what time would each have done it alone? *Ans.* A in $53\frac{4}{13}$, B in $26\frac{8}{53}$, and C in $18\frac{72}{73}$.

Ex. 32. A sets out on a journey, and travels at the rate of 25 miles per day. When he has been gone 10 days, B starts in pursuit, and travelling each day 10 miles further than he did the day preceding, overtakes A in 10 days. How far did B go the first day? *Ans.* 5 miles.

Ex. 33. Four numbers in arithmetical progression, whose sum is 32, and the sum of their squares 336. What are the numbers? *Ans.* 2, 6, 10, 14.

Ex. 34. The sum of five numbers in arithmetical progression is 160, and the sum of the square roots of the extremes is 12. What are the numbers? *Ans.* 16, 32, 48, 64.

Ex. 35. The sum of the square roots of the means of four numbers in arithmetical progression is 19, and the difference of the extremes 171. What are the numbers?

Ans. 7, 64, 121, 178.

Ex. 36. There are four numbers in arithmetical progression, such that the difference of the extremes is to the sum of the means as 3 to 11. The sum of the first and third is 30. What are the numbers? *Ans.* 12, 15, 18, 21.

Ex. 37. There are five numbers in arithmetical progression, whose sum is 40, and the sum of the square roots of the first and last 12. What is the common difference?

Ans. $6\sqrt{-26}$.

The question is therefore impossible.

Ex. 38. There are six numbers in arithmetical progression. The sum of the second and fifth is 148, and the difference of the square roots of the extremes is 10. What are the numbers? *Ans.* 4, 32, 60, 88, 116, 144.

Ex. 39. There are four numbers in arithmetical progression, which, being increased by 2, 3, 9, and 25, respectively become in geometrical progression. What are the numbers?

Ans. 3, 7, 11, 15.

Ex. 40. There are two numbers in the ratio of 2 to 3, and their sum is to the sum of their squares as 5 to 78. What are they? *Ans.* 12 and 18.

Ex. 41. A certain number is equal to 4 times the sum of its digits, and if 18 be added to it, its digits will be inverted. What is the number? *Ans.* 24.

Ex. 42. A traveller states that he has, during the last week, travelled 1326 miles; and that he has gone $2\frac{1}{2}$ times as far in steamboats as in stages, and $\frac{1}{3}$ as far in railroad cars as in steamboats. How many miles has he travelled in each of the three ways?

Ans. In stages 306 miles, in steamboats 765 miles, and in cars 255 miles.

Ex. 43. A butcher slaughtered $\frac{1}{5}$ of his sheep, and then bought 4 more; he then killed $\frac{1}{4}$ of what he had, and bought 3; after this he killed $\frac{1}{6}$ of what he then had; after which he has but 20 left. How many had he at first?

Ans. 30.

Ex. 44. A and B can finish a piece of work in 15 days, but after working 6 days, B was taken sick, and A finished it in 30 days. In what time would either have done it alone? *Ans.* A in 50 days, and B in $21\frac{3}{7}$ days.

Ex. 45. A farmer has mixed a certain number of bushels of corn and oats. Had he had 6 bushels more of each, there would have been 7 bushels of corn to every 6 of oats; but if there had been 6 bushels less of each, then he would have had 6 bushels of corn for every 5 of oats. How many were there of each? *Ans.* 78 of corn and 66 of oats.

Ex. 46. Divide 198 into five such parts, that the first increased by 1, the second by 2, the third diminished by 3, the fourth multiplied by 4, and the fifth divided by 5, may all be equal. *Ans.* 23, 22, 27, 6, and 120.

Ex. 47. A certain number consists of two digits, and is equal to the difference of the squares of its digits. If 36 be added to it, the sum will be expressed by the same digits in an inverted order. What is the number? *Ans.* 48.

Ex. 48. There are four numbers such, that the first multiplied by the sum of the other three is equal to 26; the second multiplied by the sum of the others is equal to 36; the third multiplied by the sum of the others is equal to 44; and the fourth multiplied by the sum of the others is equal to 54. What are the numbers? *Ans.* 2, 3, 4 and 6.

Ex. 49. A grocer mixed tea which cost 75 cents per pound with some which cost 55 cents per pound, and sold the whole for \$77.62½, gaining thereby 12½ per cent. How many pounds were there of each sort, the whole number being 100 lbs? *Ans.* 70 lbs. at 75, and 30 at 55 cts.

Ex. 50. A certain number, consisting of two places of figures, is equal to seven times the sum of its digits, and if 18 be subtracted from it, the digits will be inverted. What is the number? *Ans.* 42.

Ex. 51. There are four numbers in arithmetical progression such, that the product of the extremes is 36, and that of the means 54. What are the numbers?

Ans. 3, 6, 9, 12.

Ex. 52. There are four numbers in arithmetical progression, whose common difference is three times the first number, and whose sum is 44. What are the numbers?

Ans. 2, 8, 14, 20.

Ex. 53. A and B commenced trade with equal capital. A gained 25 per cent. of his stock, and B lost a sum which was 2500 dollars more than A had gained, when it was found that A's money was double B's. What was their capital? *Ans.* \$20,000.

Ex. 54. There are 4 numbers, such that if the first be increased by 1, and the last diminished by 2, they will be in arithmetical progression: their sum is 29. Required the numbers. *Ans.* 3, 6, 8, 12.

Ex. 55. The sum of the first and third of four numbers in geometrical progression is 60, and that of the second and fourth 180. Required the numbers.

Ans. 6, 18, 54, 162.

Ex. 56. Some smugglers found a cave that would exactly hold their cargo, consisting of 13 bales and 33 casks; while they were unloading, a revenue cutter appeared, on which they sailed away with 9 casks and 5 bales, having filled $\frac{2}{3}$ of the cave. How many bales, or how many casks, would the cave contain? *Ans.* 24 bales, or 72 casks.

Ex. 57. Hiero, King of Syracuse, having ordered his jeweller to make him a crown of gold, and suspecting that he had put in some silver, directed Archimedes to examine it. When weighed in water, it was found to lose $1\frac{1}{4}$ lbs. Required the number of pounds of silver it contained, the specific gravity of gold being 19·64, and of silver 10·5, and the weight of the crown 20 lbs. *Ans.* 5·22 lbs.

Ex. 58. Wishing to obtain the specific gravity of a mineral lighter than water, I first found its weight to be 15 oz. Then, having attached it to a mass of lead weighing 20 oz., the whole was found to lose 21 ounces when weighed in water. Required the specific gravity of the mineral, that of lead being 11·5. *Ans.* ·779 nearly.

Ex. 59. A piece of alloy weighing C pounds, of the specific gravity of c, is composed of two metals, A and B, whose specific gravities are a and b, respectively. How many pounds of each does it contain?

$$\textit{Ans.}\ \frac{a\ (b-c)}{c\ (b-a)}\ \text{C pounds of A,}$$

$$\text{and, } \frac{b\ (c-a)}{c\ (b-a)}\ \text{C pounds of B.}$$

Ex. 60. A and B engage in trade. A gains \$1500 and B loses \$500, when A's money is to B's as 3 to 2; but, had A lost \$500, and B gained \$1000, then A's would have been to B's as 5 to 9. What was the stock of each?

Ans. A's \$3000 and B's \$3500.

SECTION III.

Pure Equations and others which can be solved without completing the Square.

126. The only difficulty in solving equations of this class consists in reducing them so that the power may stand by itself. This being often difficult, we have appended a number of examples, the solution of which will bring before the student most of the methods which are employed in such cases. As he will probably require very frequent assistance

from his teacher on first passing over the subject, he should be required to review his work until the methods have become thoroughly familiar. He will thus have acquired a knowledge of analysis which will be of great service, not only in pursuing the remainder of the work, but also in the study of the higher branches of pure mathematics.

Ex. 1. Given $a^2+2ax+x^2=b^2$, to find the value of x.

Here the first member being a square, we extract the square root, and obtain

$$a+x=\pm b,$$

whence $$x=\pm b-a.$$

Ex. 2. Given $\sqrt{\left(\frac{x+a}{x}\right)}+2\sqrt{\left(\frac{a}{x+a}\right)}=b^2\sqrt{\left(\frac{x}{x+a}\right)}$ to find the value of x.

Multiplying by $\sqrt{x.(x+a)}$ to clear of fractions, we have

$$x+a+2\sqrt{ax}=b^2x,$$

Extract the root, and $\sqrt{x}+\sqrt{a}=\pm b\sqrt{x}$,

whence $$\sqrt{x}=\frac{\sqrt{a}}{-1\pm b},$$

and $$x=\frac{a}{(-1\pm b)^2}.$$

Ex. 3. Given $\left.\begin{array}{l} x^2y+xy^2=180 \\ x^3+y^3\ \ =189 \end{array}\right\}$ to find the values of x and y.

Adding three times the first to the second, we have

$$x^3+3x^2y+3xy^2+y^3=729,$$

whence, extracting the cube root,

$$x+y=9.$$

This divided into the first, gives

$$xy=20.$$

But, $$x^2+2xy+y^2=81,$$

and $$4xy=80,$$

whence $$x^2-2xy+y^2=1,$$

and $$x-y=\pm 1.$$

But $$x+y=9,$$

$\therefore$ $$2x=10 \text{ or } 8,$$

and $$2y=8 \text{ or } 10.$$

Hence $$x=5 \text{ or } 4,$$

and $$y=4 \text{ or } 5.$$

Ex. 4. Given $x - y = 6$ and $\left.\begin{matrix} \\ \dfrac{x^2}{y} - \dfrac{y^2}{x} = 21 \end{matrix}\right\}$ to find the values of x and y.

Clearing the last of fractions, we have

$$x^3 - y^3 = 21\,xy,$$

whence, by dividing by the first,

$$x^2 + xy + y^2 = \frac{7}{2}\,xy.$$

But $$x^2 - 2\,xy + y^2 = 36,$$

$\therefore$ $$3\,xy = \frac{7}{2}\,xy - 36,$$

and $$xy = 72,$$

also, $$x^2 + xy + y^2 = \frac{7}{2}xy = 252,$$

$\therefore$ $$x^2 + 2\,xy + y^2 = 324,$$

and $$x + y = 18,$$

from this, and $$x - y = 6$$

we obtain $$x = 12$$

and $$y = 6.$$

Ex. 5. Given $x^2 + xy + y^2 = 93$ and $x^4 + x^2y^2 + y^4 = 3441$ to find the values of x and y.

Dividing the first into the second,

$$x^2 - xy + y^2 = 37,$$

whence $$2\,xy = 56$$

and $$xy = 28,$$

$\therefore$ $$x^2 + 2\,xy + y^2 = 121$$

$$x^2 - 2\,xy + y^2 = 9,$$

Consequently $x + y = 11$ and $x - y = 3$, whence $\begin{cases} x = 7 \\ y = 4, \end{cases}$

Ex. 6. Given $\dfrac{\sqrt{x} + \sqrt{(x-a)}}{\sqrt{x} - \sqrt{(x-a)}} = \dfrac{n^2a}{x-a}$, to find the value of x.

If we multiply the numerator and denominator of the first member of the equation, by $\sqrt{x} + \sqrt{x-a}$, we will have

$$\frac{(\sqrt{x} + \sqrt{(x-a)})^2}{a} = \frac{n^2a}{x-a},$$

whence $(\sqrt{x}+\sqrt{x-a})^2=\frac{n^2a^2}{x-a}$;

Extracting the square root, $\sqrt{x}+\sqrt{x-a}=\pm\frac{na}{\sqrt{(x-a)}}$,

Clearing of fractions $\sqrt{x^2-ax}+x-a=\pm na$

$\therefore \quad \sqrt{x^2-ax}=a\pm na-x=(1\pm n)a-x,$

and $\quad x^2-ax=(1\pm n)^2a^2-2(1\pm n)ax+x^2$

whence $\quad (1\pm 2n)x=(1\pm n)^2a$

$\therefore \quad x=\frac{(1\pm n)^2a}{1\pm 2n}.$

Ex. 7. Given $\sqrt{\left(\frac{x^2+3b^2}{4}\right)}-\sqrt{\left(\frac{x^2-3b^2}{4}\right)}=\sqrt{\frac{bx^2}{c}}$.

If we multiply each member of the equation by

$\sqrt{\left(\frac{x^2+3b^2}{4}\right)}+\sqrt{\left(\frac{x^2-3b^2}{4}\right)}$, remembering that the difference multiplied by the sum gives the difference of the squares, we will have

$$\frac{x^2+3b^2}{4}-\frac{x^2-3b^2}{4}=x\sqrt{\frac{b}{c}}\left\{\sqrt{\left(\frac{x^2+3b^2}{4}\right)}+\sqrt{\left(\frac{x^2-3b^2}{4}\right)}\right\}$$

or $$\frac{3b^2}{2}=x\sqrt{\frac{b}{c}}\left\{\sqrt{\left(\frac{x^2+3b^2}{4}\right)}+\sqrt{\left(\frac{x^2-3b^2}{4}\right)}\right\}$$

whence $$\sqrt{\left(\frac{x^2+3b^2}{4}\right)}+\sqrt{\left(\frac{x^2-3b^2}{4}\right)}=\frac{3b^2}{2x}\sqrt{\frac{c}{b}}.$$

Adding the first equation, we have

$$2\sqrt{\left(\frac{x^2+3b^2}{4}\right)}=\frac{3b^2}{2x}\sqrt{\frac{c}{b}}+x\sqrt{\frac{b}{c}},$$

squaring $$x^2+3b^2=\frac{9b^3c}{4x^2}+3b^2+\frac{x^2b}{c}.$$

Whence, cancelling $3b^2$, and clearing of fractions, we have

$$4cx^4=9b^3c^2+4bx^4$$

and $$x^4=\frac{9b^3c^2}{4(c-b)}=\frac{9b^2c^2}{4}\left(\frac{b}{c-b}\right),$$

whence $$x=\sqrt{\frac{3bc}{2}}\sqrt[4]{\left(\frac{b}{c-b}\right)}.$$

Ex. 8. Given $xy = 320$ and $x^3 - y^3 : (x-y)^3 :: 61 : 1$ to find the values of x and y.

Dividing the 1st and 2d terms of the proportion by $x - y$, it becomes

$$x^2 + xy + y^2 : x^2 - 2xy + y^2 :: 61 : 1,$$

whence $x^2 + xy + y^2 = 61x^2 - 122xy + 61y^2$,

or $60x^2 + 60y^2 = 123xy = 39360$.

$\therefore$ $x^2 + y^2 = 656$,

but $2xy = 640$,

whence $x^2 + 2xy + y^2 = 1296$,

and $x^2 - 2xy + y^2 = 16$.

Consequently $x + y = 36$ and $x - y = 4$ whence $x = 20$, $y = 16$.

Ex. 9. Given $(x^2 - y^2) \times (x - y) = 3xy$ and $(x^4 - y^4) \times (x^2 - y^2) = 45x^2y^2$ to find the values of x and y.

Dividing the 2d by the 1st, we have

$$(x^2+y^2) \times (x+y) = 15xy,$$

or $x^3 + x^2y + xy^2 + y^3 = 15xy$,

but from 1st $x^3 - x^2y - xy^2 + y^3 = 3xy$, (A)

whence $2x^2y + 2xy^2 = 12xy$,

or $x + y = 6$.

Dividing this into (A), we have

$$x^2 - 2xy + y^2 = \frac{1}{2}xy, \quad \text{(B)}$$

but $x^2 + 2xy + y^2 = 36$,

$\therefore$ $4xy = 36 - \frac{1}{2}xy$,

and $xy = 8$,

whence (B) $x^2 - 2xy + y^2 = 4$,

and $x - y = 2$.

But $x + y = 6$,

consequently $x = 4$ and $y = 2$.

Ex. 10. Given $\left.\begin{aligned} x^2+x\sqrt[3]{xy^2}&=208 \\ y^2+y\sqrt[3]{x^2y}&=1053 \end{aligned}\right\}$ to find the values of x and y.

Assume $x=vy$. This substituted in the equations they become $v^2y^2+vy^2\sqrt[3]{v}=208$,

and $y^2+y^2\sqrt[3]{v^2}=1053$.

Dividing the latter by the former, we have

$$\frac{v^2+v\sqrt[3]{v}}{1+\sqrt[3]{v^2}} \text{ or } \frac{\sqrt[3]{v^4}+\sqrt[3]{v^6}}{1+\sqrt[3]{v^2}} \text{ or } \sqrt[3]{v^4}=\frac{16}{81},$$

whence $v=\frac{x}{y}=\frac{8}{27}$ and $x=\frac{8}{27}y$.

This substituted in the 1st equation, gives

$$\frac{64}{729}y^2+\frac{8}{27}y\sqrt[3]{\frac{8}{27}y^3}=208,$$

or $\frac{64}{729}y^2+\frac{16}{81}y^2$, that is $\frac{208}{729}y^2=208$,

$\therefore$ $y^2=729$ and $y=\pm 27$,

consequently $x=\frac{8}{27}y=\pm 8$.

2d Solution. The equations may be written

$$\sqrt[3]{x^4}(\sqrt[3]{x^2}+\sqrt[3]{y^2})=208,$$

and $\sqrt[3]{y^4}(\sqrt[3]{x^2}+\sqrt[3]{y^2})=1053$,

whence by division $\sqrt[3]{\frac{x^4}{y^4}}=\frac{208}{1053}=\frac{16}{81}$,

and $\frac{x}{y}=\frac{8}{27}$ as before.

Ex. 11. Given $\left.\begin{aligned} x-y&=4 \\ xy&=45 \end{aligned}\right\}$ to find the values of x and y.

Ans. $x=9$, $y=5$.

Ex. 12. Given $\left.\begin{aligned} x+y&=10 \\ xy&=21 \end{aligned}\right\}$ to find the values of x and y.

Ans. $x=7$, $y=3$.

Ex. 13. Given $\left.\begin{aligned} 5x+3y&=66 \\ xy&=63 \end{aligned}\right\}$ to find the values of x and y.

Ans. $x=9$, $y=7$.

Ex. 14. Given $\left.\begin{array}{l} 7x + \frac{1}{7}y = 23 \\ xy \quad = 42 \end{array}\right\}$ to find the values of x and y.

Ans. $x = 3$, $y = 14$.

Ex. 15. Given $\left.\begin{array}{l} ax + by = c \\ xy \quad = d \end{array}\right\}$ to find the values of x and y.

Ans. $x = \frac{c + \sqrt{c^2 - 4\,abd}}{2\,a}$

$y = \frac{c - \sqrt{c^2 - 4\,abd}}{2\,b}$.

Ex. 16. Given $\left.\begin{array}{l} x + y : x - y :: a : b \\ xy = c^2 \end{array}\right\}$ to find the values of x and y. *Ans.* $x = \pm c\sqrt{\left(\frac{a+b}{a-b}\right)}$, $y = \pm c\sqrt{\left(\frac{a-b}{a+b}\right)}$.

Ex. 17. Given $\left.\begin{array}{l} x^2 - xy = 21 \\ xy - y^2 = 12 \end{array}\right\}$ to find the values of x and
and y.

Ans. $x = 7$, $y = 4$.

Ex. 18. Given $\left.\begin{array}{l} x^2 + y^2 : x^2 - y^2 :: a : b \\ xy^2 = c^3 \end{array}\right\}$ to find the values of x and y. *Ans.* $x = c\sqrt[3]{\left(\frac{a+b}{a-b}\right)}$, $y = c\sqrt[6]{\left(\frac{a-b}{a+b}\right)}$.

Ex. 19. Given $\sqrt{\frac{a+x}{a-x}} + \sqrt{\frac{a-x}{a+x}} = b$, to find the value of x.

Ans. $x = \pm\frac{a}{b}\sqrt{b^2 - 4}$.

Ex. 20. Given

$\sqrt{(x + \sqrt{x})} - \sqrt{(x - \sqrt{x})} = \frac{3}{2}\sqrt{\left(\frac{x}{x + \sqrt{x}}\right)}$, to find the value of x.

Ans. $x = \frac{25}{16}$.

Ex. 21. Given $\frac{\sqrt{x+a} + \sqrt{x-a}}{\sqrt{x+a} - \sqrt{x-a}} = b$, to find the value of x.

Ans. $x = \frac{a}{2b}(1 + b^2.)$

Ex. 22. Given $\frac{a}{a-\sqrt{a^2-x^2}}-\frac{a}{a+\sqrt{a^2-x^2}}=\frac{\sqrt{b}}{x^2}$, to find the value of x.

Ans. $x=\pm\frac{1}{2a}\sqrt{4a^4-b}$.

Ex. 23. Given $\frac{\sqrt{a}-\sqrt{a-x}}{\sqrt{a}+\sqrt{a-x}}=a$, to find the value of x.

Ans. $x=\left(\frac{2a}{a+1}\right)^2$.

Ex. 24. Given $\frac{\sqrt{4x+1}+\sqrt{4x}}{\sqrt{4x-1}-\sqrt{4x}}=9$, to find the value of x.

Ans. $x=\frac{4}{9}$.

Ex. 25. Given $\frac{1}{x}+\frac{1}{a}=\sqrt{\left\{\frac{1}{a^2}+\sqrt{\left(\frac{4}{a^2x^2}+\frac{9}{x^4}\right)}\right\}}$ to find the value of x.

Ans. $x=2a$.

Ex. 26. Given $\left.\begin{array}{l}x^2+xy=60\\ xy+y^2=84\end{array}\right\}$ to find the values of x and y.

Ans. $x=5$, $y=7$.

Ex. 27. Given $\sqrt{\left(\frac{a^2}{x^2}+b^2\right)}-\sqrt{\left(\frac{a^2}{x^2}-b^2\right)}=b$; to find the value of x.

Ans. $x=\pm\frac{2a}{b\sqrt{5}}$.

Ex. 28. Given $\left.\begin{array}{l}x^2-xy=48y\\ xy-y^2=3x\end{array}\right\}$ to find the values of x and y.

Ans. $x=16$ or $-\frac{48}{5}$.

$y=4$ or $\frac{12}{5}$.

Ex. 29. Given $\sqrt{\frac{x+a}{x}}+2\sqrt{\frac{a}{x+a}}=b^2\sqrt{\frac{x}{x+a}}$, to find the value of x.

Ans. $x=\frac{a}{(b\pm 1)^2}$.

Ex. 30. Given $\left.\begin{array}{l}x^2+y\sqrt{xy}=9\\ y^2+x\sqrt{xy}=18\end{array}\right\}$ to find the values of x and y.

Ans. $x=\pm 1$, $y=\pm 4$.

Ex. 31. Given $\left.\begin{array}{r}(x^2+y^2)(x+y)=2336\\(x^2-y^2)(x-y)=576\end{array}\right\}$ to find the values of x and y. *Ans.* $x = 11$ or 5
$y = 5$ or 11.

Ex. 32. Given $\left.\begin{array}{r}x^3+y^3 : x^3-y^3 :: 559 : 127\\x^2y=294\end{array}\right\}$ to find the values of x and y. *Ans.* $x=7$, $y=6$.

Ex. 33. Given $\left.\begin{array}{r}x^2+y^2+xy(x+y)=68\\x^3+y^3-3x^2=12+3y^2\end{array}\right\}$ to find the values of x and y. *Ans.* $x=4$ or 2
$y=2$ or 4.

Ex. 34. Given $\left.\begin{array}{r}\frac{9}{8}\frac{\sqrt[3]{x+y}}{y}+\frac{9}{8}\frac{\sqrt[3]{x+y}}{x}=\frac{8}{7}\\ \frac{7}{4}\frac{\sqrt[3]{x-y}}{y}-\frac{7}{4}\frac{\sqrt[3]{x-y}}{x}=\frac{1}{9}\end{array}\right\}$ to find the values of x and y. *Ans.* $x=\frac{9}{2}$, $y=\frac{7}{2}$.

127. *Problems producing Pure Equations.*

EXAMPLES.

Ex. 1. It is required to divide the number 24 into two parts, whose squares shall be as 25 to 9.

Let x = the greater part,
then $24-x$ = the less part,
and $x^2 : (24-x)^2 :: 25 : 9$,
whence $x : 24-x :: 5 : 3$.
Or $3x = 120-5x$,
$\therefore$ $\left.\begin{array}{r}x=15\\24-x=9\end{array}\right\}$ the parts required.

Ex. 2. The number of square feet in a right-angled triangle is equal to the number of feet in its three sides, and the square of the number of feet in the hypothenuse is less than the square of the sum of the other two sides, by half the product of the number of square feet in the area, by the number of feet in the base. Required the three sides.

NOTE.—The square upon the hypothenuse is equal to the sum of the squares on the other sides, and the area is equal to half the product of those sides.

Let $x =$ the number of feet in the base,
and $y =$ the number in the altitude,
then $\sqrt{x^2+y^2} =$ the number in the hypothenuse,
and $\frac{xy}{2} =$ the number of square feet in the area.

$$\therefore \quad \frac{xy}{2} = x + y + \sqrt{x^2+y^2},$$

and $$x^2 + y^2 = x^2 + 2\,xy + y^2 - \frac{x^2y}{4},$$

by transposition $$\frac{x^2y}{4} = 2\,xy,$$

and $$x = 8.$$

Hence, from the first equation $4\,y = 8 + y + \sqrt{64 + y^2}$,
wherefore $3\,y - 8 = \sqrt{64 + y^2}$,
and squaring $9\,y^2 - 48\,y + 64 = 64 + y^2$,
$\therefore$ $8\,y = 48$,
and $y = 6$.
Consequently the hypothenuse $= 10$.
$\therefore$ the sides are 6, 8, and 10 feet respectively.

Ex. 3. A farmer has two cubical stacks of hay, of which one contains 117 cubic yards more than the other. Required the dimensions of each, the side of the larger being 3 yards longer than that of the other.

Let $x =$ the number of yards in the side of the other; and $y =$ the number in the side of the smaller,

then $x - y = 3$
and $x^3 - y^3 = 117$,
dividing the second by the first, $x^2 + xy + y^2 = 39$,
and squaring the first $x^2 - 2\,xy + y^2 = 9$,
$\therefore$ by subtraction $3\,xy = 30$,
and $xy = 10$,

now $x^2 + xy + y^2 = 39,$
∴ by addition $x^2 + 2xy + x^2 = 49,$
whence $x + y = 7,$
but $x - y = 3,$
∴ $x = 5$ and $y = 2.$

Ex. 4. There are four numbers in geometrical progression, such that the sum of the extremes is 56 and the sum of the means 24. What are the numbers?

Here let x and y represent the means,

then the extremes will be $\frac{x^2}{y}$ and $\frac{y^2}{x}$,

∴ $x + y = 24$

$$\frac{x^2}{y} + \frac{y^2}{x} = 56.$$

Clearing the last of fractions,

$$x^3 + y^3 = 56\,xy,$$

cubing the first $x^3 + 3x^2y + 3xy^2 + y^3 = 13824,$
∴ by subtraction $3\,x^2y + 3\,xy^2 = 13824 - 56\,xy,$

dividing this by the first equation $3\,xy = 576 - \frac{7}{3}xy,$

clearing of fractions $9\,xy = 1728 - 7\,xy,$
whence $xy = 108.$
But from the first $x^2 + 2\,xy + y^2 = 576$
and $4\,xy = 432.$
∴ by subtraction $x^2 - 2\,xy + y^2 = 144,$
and $x - y = 12.$
But $x + y = 24,$
∴ $x = 18,\ y = 6,$

$$\frac{x^2}{y} = 54,\quad \frac{y^2}{x} = 2,$$

∴ 54, 18, 6 and 2 are the numbers required.

Ex. 5. A person has two pieces of land, one in the form of a right-angled triangle, and the other in that of a rectangle, the longer side of which is equal to the hypothenuse of the triangle, and the other to half the greater side; but wishing to have his land in one piece, he exchanged for a square piece of equal area, whose side was twice as long as the shorter side of the rectangle. By this exchange he has saved 25 poles of fencing. What are the areas of the triangle and rectangle, and what is the length of each of their sides?

Let $2x =$ length of the greater side of the triangle, and $y =$ that of the less;

then $\sqrt{4x^2 + y^2} =$ that of the hypothenuse,

$\therefore$ $xy =$ the area of the triangle,

and $x\sqrt{4x^2 + y^2} =$ the area of the rectangle;

also $2x =$ the side of the square,

$\therefore$ $4x^2 = xy + x\sqrt{4x^2 + y^2}$,

or $4x = y + \sqrt{4x^2 + y^2}$. (A)

Again $8x =$ the perimeter of the square,

$\therefore$ $8x + 25 = 2x + y + \sqrt{4x^2 + y^2} + 2x + 2\sqrt{4x^2 + y^2}$,

or $4x + 25 = y + 3\sqrt{4x^2 + y^2}$,

but (A) $12x = 3y + 3\sqrt{4x^2 + y^2}$,

$\therefore$ $8x - 25 = 2y$,

and by transposition $8x - 2y = 25$,

but from (A) $8x - 2y = 2\sqrt{4x^2 + y^2}$,

$\therefore$ $2\sqrt{4x^2 + y^2} = 25$

and $16x^2 + 4y^2 = 625$;

and substituting the value of $2y$, obtained above,

$$16x^2 + 64x^2 - 400x + 625 = 625,$$

whence $80x = 400$

and $x = 5$,

also $y = \dfrac{8x - 25}{2} = 7\frac{1}{2}$.

$\therefore$ the sides of the triangle are 10, $7\frac{1}{2}$ and $12\frac{1}{2}$ rods; the sides of the rectangle $12\frac{1}{2}$ and 5 rods; and the areas of the triangle and rectangle, $37\frac{1}{2}$ and $62\frac{1}{2}$ square rods respectively.

Ex. 6. It is required to divide the number 14 into two such parts that the quotient of the greater divided by the less, may be to the quotient of the less by the greater as 16 to 9. *Ans.* 8 and 6.

Ex. 7. Bought a number of oxen for 1406·25 dollars, the number of dollars per head being to the number of oxen as 9 to 4. How many did he buy, and what did he give for each? *Ans.* 25 oxen, at $56·25 per head.

Ex. 8. There are two numbers whose sum is to the less as 90 is to the greater, and whose sum is to the greater as 40 is to the less. What are the numbers?

Ans. 36 and 24.

Ex. 9. The sides of a rectangle are to each other as 5 to 7, and its area is 26 A. 1 r. 35 p. How many rods are there on each side? *Ans.* 55 and 77 rods.

Ex. 10. A has a rectangular tract of land, the four sides of which measure 336 rods, from which he sells a rectangular portion containing 2 A. 3 r. 25 p. Required the dimensions of the smaller piece, its length being $\frac{1}{3}$ and its breadth $\frac{1}{4}$ of that of the whole tract; and what is the content of the whole?

Ans. Length 31 and breadth 15 rods; contents of whole tract 43 A. 2 r. 15 p.

Ex. 11. A merchant purchased two pieces of cloth, one of which cost $\frac{1}{3}$ and the other $\frac{1}{4}$ as many dollars per yard as there were yards in its length. Now, had the whole been bought at the price of the first, the cost would have been $315. But had he only paid as much per yard for the first as he did for the second, they would have cost $270. What number of yards was there in each?

Ans. 21 yards in the first, and 24 yards in second.

Ex. 12. There are two numbers whose sum is 40, and the difference of whose squares is equal to 4 times the square of their difference. What are the numbers?

Ans. 25 and 15.

Ex. 13. A and B engaged to work for a certain number of days. At the end of the time, A, who had been absent 4 days, received $18·75, while B, who had been absent 7 days, received only $12. Now, had B been absent 4, and A 7 days, each would have been entitled to the same sum. For how many days were they engaged, and at what rate?

Ans. They were engaged to work 19 days, and A received $1·25 and B $1·00 per day.

Ex. 14. A and B have two rectangular tracts of land, their lengths being as 7 to 6, and difference between their areas 150 A., B's being the greater. Now, had A's been as broad as B's, it would have been 672 rods long. But had B's been as broad as A's, it would have been 900 rods long. How many acres were there in each?

Ans. A's 2100 acres, and B's 2250 acres.

Ex. 15. A person has a cask of wine containing 256 gallons; from which he draws a certain quantity, and then fills the vessel with water. He again draws off the same quantity as before, and so on for 4 times, filling the cask with water after every draught, when there were only 81 gallons of pure wine left. How much wine did he draw each time?

Ans. 64, 48, 36 and 27 gallons.

Ex. 16. There are two numbers, whose difference multiplied by the less produces 42; but when multiplied by the sum, the product is 133. What are the numbers?

Ans. 13 and 6.

Ex. 17. Required two numbers, such that the sum of their cubes may be to the cube of their sum as 7 to 25, and the sum of their squares multiplied by the greater may be equal to 1053. *Ans.* 9 and 6.

Ex. 18. What two numbers are those whose difference multiplied by the greater makes 60, but when multiplied by the less makes 44? *Ans.* 15 and 11.

Ex. 19. A person laid out a certain sum of money upon a speculation, upon which he found he had gained £69 the first year. This he added to his stock, and at the end of the next year he found he had gained as much per cent. as in the year preceding. Proceeding in the same manner for four years, he found that at the end of the time his stock was to the sum first invested as 243 to 48. What was the sum laid out, and the gain per cent.?

Ans. Stock, £138; gain per cent., 50.

Ex. 20. The sum of three numbers in geometrical progression is 26, and the sum of their squares 364. What are the numbers? *Ans.* 2, 6, and 18.

Ex. 21. There are four numbers in geometrical progression, such that the sum of the extremes is 140, and the sum of the means 60. What are the numbers?

Ans. 5, 15, 45, and 135.

Ex. 22. Required four numbers in geometrical progression, such that the difference of the extremes may be to the difference of the means as 19 to 6, and the sum of the means may be 30. *Ans.* 8, 12, 18, and 27.

Ex. 23. The sum of two numbers multiplied by the sum of their squares, is equal to $13\frac{1}{3}$ times their product: and the sum of the squares multiplied by the difference of their fourth powers, to $88\frac{8}{9}$ times the square of the product. What are the numbers? *Ans.* 3 and 1.

Ex. 24. The difference of the extremes of four numbers in geometrical progression is $15\frac{1}{4}$, and the difference of the means 5. What are the numbers?

Ans. 16, 20, 25, and $31\frac{1}{4}$.

SECTION IV.

Adfected Quadratics.

128. Adfected quadratics are such as contain the square and first power of the unknown, with an arbitrary quantity. Of this class are

$$5x^2 + 3x = 10, \quad 7x^2 + 6 = 10x, \text{ \&c.}$$

In order to find the unknown quantities in such equations we must so arrange the left hand member that it may be a square; then, extracting the root, the equation is reduced to a simple equation, which may be solved by the rules already laid down.

129. The square on the left member may be completed in various ways; there are, however, three principal methods, either of which will apply to any case that can present itself. These rules are founded on the formulas

1. $(x+b)^2=x^2+2bx+b^2$, or $(x+\frac{1}{2}b)^2=x^2+bx+\frac{1}{4}b^2$,

2. $(2ax+b)^2=4a^2x^2+4abx+b^2$,

3. $(ax+b)^2=a^2x^2+2abx+b^2$.

If then we have an expression similar to ax^2+bx, it may be rendered a complete square by either of the following methods, viz.,

1st. *Divide by the coefficient of x^2, and then add the square of half the coefficient of x in the quotient.* This rule evidently changes

$$ax^2+bx, \text{ into } x^2+\frac{bx}{a}+\frac{b^2}{4a^2}=\left(x+\frac{b}{2a}\right)^2.$$

2d. *Multiply by the coefficient of x^2, and to the product add the square of half the coefficient of x in the original expression.* This evidently changes

$$ax^2+bx \text{ into } a^2x^2+a^2bx+\frac{b^2}{4}=\left(ax+\frac{b}{2}\right)^2.$$

3d. *Multiply by 4 times the coefficient of x^2, and add the square of the original coefficient of x.* This changes

$$ax^2+bx \text{ into } 4a^2x^2+4abx+b^2=(2ax+b)^2.$$

Either of these rules, as has been remarked, will apply to every case. The one which it will be most convenient to employ will depend upon the coefficients of the different terms.

Examples.

Ex. 1. Given $x^2-6x=40$. In this case we shall use method 1; adding $3^2=9$ to each member, the equation becomes $x^2-6x+9=49$.

Extracting the square root, $x-3=\pm 7$,

whence $x=10$ or -4.

Ex. 2. Given $3x^2 - 2x = 408$, to find the value of x.

Here, by method 2, we have

$$9x^2 - 6x + 1 = 1225,$$

whence $3x - 1 = \pm 35,$

and $x = 12$ or $-11\frac{1}{3}$.

Ex. 3. Given $5x^2 - 27x + 70 = 36$, to find the value of x.

Transposing $5x^2 - 27x = -34.$

Whence, by method 3, we have

$$100x^2 - 540x + 729 = 49,$$

$\therefore$ $10x - 27 = \pm 7$

$10x = 34$ or 20,

and $x = 3\frac{2}{5}$ or 2,

Ex. 4. Given $ax^2 - 2bx = c$, to find the value of x.

Here, by method 2, we have

$$a^2x^2 - 2abx + b^2 = ac + b^2,$$

whence $ax - b = \pm\sqrt{ac + b^2},$

and $ax = b \pm \sqrt{ac + b^2},$

$\therefore$ $x = \frac{b}{a} \pm \frac{1}{a}\sqrt{ac + b^2}.$

Ex. 5. Given $5x + \frac{18 - x}{x} = 59 + \frac{x + 6}{x}$, to find the value of x.

Clearing of fractions and transposing, we have

$$5x^2 - 61x = -12.$$

Completing the square, (Method 3,)

and $100x^2 - 1220x + 3721 = 3481,$

whence $10x - 61 = \pm 59,$

and $x = 12$ or $\frac{1}{5}$.

Ex. 6. Given $5x^2 - 27 = 20x + 33$, to find the values of x. *Ans.* 6 or -2.

Ex. 7. Given $5x-\frac{25-3x}{x}=23$, to find the values of x. *Ans.* 5 or -1.

Ex. 8. Given $\frac{3x}{x-8}+\frac{7x-5}{2x+3}=-18\frac{7}{17}$, to find the values of x. *Ans.* 7 and $-\frac{976}{847}$.

Ex. 9. Given $\frac{27}{18x-x^2}+\frac{54}{x^2+6x}=\frac{33}{5x}$, to find the values of x. *Ans.* 9 or $\frac{78}{11}$.

Ex. 10. Given $14+4x-\frac{x+7}{x-7}=3x+\frac{9+4x}{3}$, to find the values of x. *Ans.* 9 and 28.

Ex. 11. Given $\frac{x}{x+60}=\frac{7}{3x-5}$, to find the values of x. *Ans.* 14 and -10.

Ex. 12. Given $\sqrt{x^5}-\frac{40}{\sqrt{x}}=3x$, to find the values of x. *Ans.* 4 or $(-5)^{\frac{2}{3}}$.

Ex. 13. Given $x+5-\sqrt{x+5}=6$, to find the values of x.

Assume $\sqrt{x+5}=y$, and we have

$$y^2-y=6.$$

Whence

$$y=3 \text{ or } -2,$$

and $$x+5=y^2=9 \text{ or } 4,$$

$\therefore$ $$x=4 \text{ or } -1.$$

The last value, if substituted in the equation, gives

$$4-\sqrt{4}=6,$$

which at first may appear incorrect. In this case, however,

$$\sqrt{x+5}=\sqrt{4}=-2,$$

the minus sign being given to it in accordance with the value of y.

130. We not unfrequently have cases similar to the above, in which one of the values requires to be taken with some limitation, and sometimes there are new values introduced in the course of the operation, which, though they satisfy the equation from which they are immediately derived, will not satisfy the original equation. It is therefore necessary, when the solution has been a complex one, to test the results by substitution.

Ex. 14. Given $x+16-7\sqrt{x+16}=10-4\sqrt{x+16}$, to find the values of x.

By transposition we have

$$x+16-3\sqrt{x+16}=10.$$

If we consider $\sqrt{x+16}$ as the unknown, we will have, by completing the square,

$$4\ (x+16)-12\sqrt{x+16}+9=49,$$

whence $\quad 2\sqrt{(x+16)}-3=\pm 7,$

and $\quad x=9 \text{ or } -12.$

Ex. 15. Given $x^2+5\sqrt{x^2-16x}=16x+300$, to find the values of x. *Ans.* 25, -9 and $8\pm 4\sqrt{29}$.

Ex. 16. Given $3x^2-2x+5\sqrt{6x^2-4x+1}=6x^2-4x+5$, to find the values of x.

Ans. 4, $\frac{2}{3}$, $\frac{-10}{3}$ and 0.

Ex. 17. Given $(x+6)^2+2x^{\frac{1}{2}}(x+6)=138+x^{\frac{1}{2}}$, to find the values of x.

Ans. 4, 9 and $\frac{-33\pm\sqrt{-67}}{2}$.

NOTE. In this example, consider $x+6$ the unknown, and complete the square without transposition.

Ex. 18. Given $x^4-2x^3+x=132$, to find the values of x.

In this case if we endeavour to extract the square root, we shall find said root to be $x^2 - x$, with a remainder $-x^2 + x$. The original expression is therefore equivalent to

$$(x^2 - x)^2 - (x^2 - x) = 132.$$

Completing the square

$$4(x^2 - x)^2 - 4(x^2 - x) + 1 = 529,$$

whence $\quad 2(x^2 - x) - 1 = \pm 23,$

and $\quad x^2 - x = 12 \text{ or } -11,$

$\therefore \quad 4x^2 - 4x + 1 = 49 \text{ or } -43,$

and $\quad x = 4, -3 \text{ or } \dfrac{1 \pm \sqrt{-43}}{2}.$

Ex. 19. Given $x^4 - 6mx^3 + 27m^3x = \frac{19}{4}m^4$, to find the values of x.

Ans. $\frac{m}{2}(3 \pm \sqrt{47})$ or $\frac{m}{2}.(3 \pm \sqrt{7})$.

Ex. 20. Given $x^2 - 2x^{\frac{3}{2}} + 2x - \sqrt{x} = 6$, to find the values of x.

Ans. 4, 1 or $\dfrac{-5 \pm \sqrt{-11}}{2}$.

Ex. 21. Given $\sqrt{x} - \frac{8}{x} = \frac{7}{\sqrt{x} - 2}$, to find the values of x.

Ans. 16 or 1.

Ex. 22. Given $\sqrt{12 - \frac{12}{x^2}} + \sqrt{x^2 - \frac{12}{x^2}} = x^2$, to find the values of x.

Ans. ± 2 or $\pm\sqrt{-3}$.

Ex. 23. Given $x^4 + \frac{13}{3}x^3 - 39x = 81$, to find the values of x.

Ans. ± 3 or $\dfrac{-13 \pm \sqrt{-155}}{6}$.

Ex. 24. Given $\dfrac{x + \sqrt{x}}{x - \sqrt{x}} + \dfrac{1}{2} = +5\dfrac{x - \sqrt{x}}{x + \sqrt{x}}$ to find the values of x.

Ans. 9 or $\frac{9}{49}$.

Here, if we multiply both members of the equation by $\frac{x+\sqrt{x}}{x-\sqrt{x}}$, we will have

$$\left\{\frac{x+\sqrt{x}}{x-\sqrt{x}}\right\}^2+\frac{1}{2}\left\{\frac{x+\sqrt{x}}{x-\sqrt{x}}\right\}=5,$$

whence, $\frac{x+\sqrt{x}}{x-\sqrt{x}}=2$ or $-\frac{5}{2}$.

Clearing of fractions, we have

$$x+\sqrt{x}=2x-2\sqrt{x} \text{ or } -\frac{5}{2}x+\frac{5}{2}\sqrt{x},$$

whence $\sqrt{x}=3$, or $\frac{3}{7}$,

$\therefore$ $x=9$, or $\frac{9}{49}$.

Ex. 25. Given $\frac{2x+\sqrt{x}}{2x-\sqrt{x}}=3\frac{7}{15}-3\frac{2x-\sqrt{x}}{2x+\sqrt{x}}$, to find the values of x. *Ans.* 4 or $\frac{49}{16}$.

Ex. 26. Given $\sqrt{x-\frac{1}{x}}+\sqrt{1-\frac{1}{x}}=x$ to find the values of x. *Ans.* $\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$.

Ex. 27. Given $\sqrt[3]{5+x}+\sqrt[3]{30-x}=5$, to find the values of x. *Ans.* 3 or 22.

Ex. 28. Given $\sqrt{x^2-\frac{a^4}{x^2}}+\sqrt{a^2-\frac{a^4}{x^2}}=\frac{x^2}{a}$, to find the values of x. *Ans.* $\pm a\sqrt{\frac{1\pm\sqrt{5}}{2}}$

Ex. 29. Given $\frac{1}{(2x-4)^2}=\frac{1}{8}+\frac{2}{(2x-4)^4}$, to find the values of x. *Ans.* 3 and 1.

Ex. 30. Given $\frac{18+x}{6(3-x)}=\frac{20x+9}{19-7x}-\frac{65}{4(3-x)}$, to find the values of x. *Ans.* $2\frac{1}{2}$ and $7\frac{22}{113}$.

Ex. 31. Given $x^{-4}-9x^{-2}+20=0$, to find the values of x.

Ans. $\pm\frac{1}{5}\sqrt{5}$ and $\pm\frac{1}{2}$.

Ex. 32. Let $x+\sqrt{x} : x-\sqrt{x} :: 3\sqrt{x}+6 : 2\sqrt{x}$. What are the values of x?

Ans. 9 and 4.

Ex. 33. Let $x^2-7x+\sqrt{x^2-7x+18}=24$, to find the values of x.

Ans. $x=9$ or -2 or $\frac{7\pm\sqrt{173}}{2}$.

Ex. 34. Given $x^2(x^2-4)^{-2}+\frac{6}{x^2-4}=\frac{351x^{-2}}{25}$, to find the values of x.

Ans. ± 3 and $\pm\frac{1}{11}\sqrt{429}$.

Ex. 35. Given $x^2\sqrt{x+4}+4x^2+16x=21x\sqrt{x+4}+84\sqrt{x+4}$, to find the values of x.

Ans. -4 12 or -3 or $\frac{49\pm\sqrt{3185}}{2}$.

Ex. 36. Given $\sqrt{a-\frac{a}{x^2}}+\sqrt{x^2-\frac{a}{x^2}}=x^2$, to find the values of x.

Ans. $\pm\sqrt{\left(\frac{1}{2}\pm\frac{1}{2}\sqrt{4a+1}\right)}$.

Ex. 37. Given $27x^2-\frac{841}{3x^2}+\frac{17}{3}=\frac{232}{3x}-\frac{1}{3x^2}+5$, to find the values of x.

Ans. 2 or $-\frac{14}{9}$ or $-\frac{2}{9}\pm\frac{1}{9}\sqrt{-266}$.

Ex. 38. Let $\sqrt{\left(\frac{x^4-a^4}{x^2}\right)}+\frac{a}{x}\sqrt{x^2-a^2}=\frac{x^2}{a}$ to find the values of x.

Ans. $\pm\frac{a}{2}\sqrt{2\pm2\sqrt{5}}$.

Ex. 39. Given

$$\frac{16-4\sqrt{x}}{8-3\sqrt{x}}=\frac{88+33\sqrt{x}}{4+\sqrt{x}}+\frac{x^2-5x+11}{(8-3\sqrt{x})(4+\sqrt{x})},$$

to find the values of x. *Ans.* 93 or 7.

Ex. 40. Given $2x^{\frac{3}{2}}(x^3+a^3)^{\frac{1}{2}}=2x^2(x+2a)+a^2(x-a)$, to find the values of x. *Ans.* $\frac{a}{2}$ or $-a$.

131. The most general form in which a quadratic containing two unknown quantities can be presented, is one which contains the squares of the unknowns, their products, and first powers, besides an absolute quantity. Such an equation is

$$ax^2+bxy+cy^2+dx+ey+f=0.$$

132. If between two such equations we eliminate one of the unknowns, the resulting equation will be of the fourth degree. We cannot, therefore, *in general*, solve equations of this nature. There are, however, many cases in which such reductions or combinations may be made as shall reduce the result to a quadratic form. The methods of performing this must depend on the ingenuity of the student. If, however, he has thoroughly mastered the preceding section, and that on pure equations, he will find comparatively little difficulty. He will also see, in the examples which follow, most of the more important artifices employed. He should study the examples that are solved, carefully, until he has made these artifices his own: his own ingenuity must then do much of the rest for him.

In all cases where one of the equations is of the first degree, the equation resulting from the elimination cannot exceed the second degree. Such equations are, therefore, readily solved.

Examples.

Ex. 1. Given $\left.\begin{array}{r}3x+y=18\\ x^2+2y^2=43\end{array}\right\}$ to find the values of x and y.

Here, from the first equation $y=18-3x$,

whence $$y^2=324-108x+9x^2.$$

Substituting this in the second equation,

we have $x^2 + 648 - 216x + 18x^2 = 43$,

whence $\quad 19x^2 - 216x = -605$,

$\therefore$ completing the square $361x^2 - 4104x + 11664 = 169$,

and $\quad 19x - 108 = \pm 13$,

whence $\quad 19x = 95$ or 121,

and $\quad x = 5$ or $\dfrac{121}{19}$.

Ex. 2. Given $\left.\begin{array}{r} 2x + y = 10 \\ 2x^2 - xy + 3y^2 = 54 \end{array}\right\}$ to find the values of x and y.

From the first equation $y = 10 - 2x$,

whence $\quad y^2 = 100 - 40x + 4x^2$.

Substituting this in the second equation, it becomes

$$2x^2 - 10x + 2x^2 + 300 - 120x + 12x^2 = 54.$$

Whence $\quad 16x^2 - 130x = -246$,

$\therefore$ completing the square

$$256x^2 - 2080x + 4225 = 289,$$

and extracting the root $16x - 65 = \pm 17$,

whence $\quad 16x = 82$ or 48,

$\therefore \quad x = \dfrac{41}{8}$ or 3,

and $\quad y = 10 - 2x = -\frac{1}{4}$ or 4.

Ex. 3. Let $\quad xy = 15$

and $\quad \dfrac{x^2}{y} + \dfrac{y^2}{x} = 10\dfrac{2}{15}$, to find the values of x and y.

Multiplying the two equations, we have

$$x^3 + y^3 = 152,$$

whence $\quad x^6 + x^3y^3 = 152x^3$,

but $\quad x^3y^3 = 3375$,

$\therefore$ by subtraction $x^6 = 152x^3 - 3375$

and $\quad x^6 - 152x^3 = -3375$,

consequently $\quad x^3 = 27$ or 125,

and $\quad x = 3$ or 5,

$\therefore \quad y = 5$ or 3.

Ex. 4. Let $x + y = 10$ and $x^4 + y^4 = 2482$ } to find the values of x and y.

Put $x = a + z$ and $y = a - z$,

then $x + y = 2a = 10$.

Also (Art. 85.) $x^4 = a^4 + 4a^3z + 6a^2z^2 + 4az^3 + z^4$,

and $y^4 = a^4 - 4a^3z + 6a^2z^2 - 4az^3 + z^4$,

$\therefore \quad x^4 + y^4 = 2a^4 + 12a^2z^2 + 2z^4 = 2482$,

and by transposition, &c., $z^4 + 6a^2z^2 = 1241 - a^4$,

or substituting the value of a

$$z^4 + 150z^2 = 616.$$

Completing the square and extracting the root

$$z^2 + 75 = \pm 79,$$

whence $z = \pm\sqrt{-154}$ or ± 2,

and $x = a + z = 5 \pm \sqrt{-154}$ or 7 or 3,

$y = a - z = 5 \mp \sqrt{-154}$ or 3 or 7.

Second Solution. From the 1st we have

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 10000$

but $x^4 + y^4 = 2482$

$\therefore$ by subtraction $4x^3y + 6x^2y^2 + 4xy^3 = 7518$.

Multiply the square of the first equation by $4xy$, and we have $4x^3y + 8x^2y^2 + 4xy^3 = 400xy$.

Subtracting this from the preceding, and transposing,

and $2x^2y^2 - 400xy = -7518$,

whence $x^2y^2 - 200xy = -3759$,

$\therefore \quad xy = 179$ or 21.

But from the 1st $x^2 + xy = 10x$,

$\therefore \quad x^2 - 10x = -179$ or -21,

and $x = 5 \pm \sqrt{-154}$, or 7 or 3,

$y = 5 \mp \sqrt{-154}$, or 3 or 7.

Ex. 5. Given $x + y = s$ and $xy = p$ } to find the values of $x^2 + y^2$, $x^3 + y^3$, $x^4 + y^4$, &c.

Squaring the first, we have $\quad x^2 + 2xy + y^2 = s^2,$

but $\quad 2xy \qquad = 2p.$

∴ by subtraction $\quad x^2 + y^2 \qquad = s^2 - 2p.$ (A)

Multiply this by the 1st and $\quad x^3 + x^2y + xy^2 + y^3 = s^3 - 2sp,$

but the product of the 1st and 2d is $\quad x^2y + xy^2 \qquad = sp,$

∴ by subtraction $\quad x^3 + y^3 \qquad = s^3 - 3sp.$ (B)

Multiply by the first, and $\quad x^4 + x^3y + xy^3 + y^4 = s^4 - 3s^2p,$

Multiply the equation (A) by the 2nd and $\quad x^3y + xy^3 \qquad = s^2p - 2p^2,$

∴ by subtraction $\quad x^4 + y^4 \qquad = s^4 - 4s^2p + 2p^2.$ (C)

Multiply again by the first, and $\quad x^5 + x^4y + xy^4 + y^5 = s^5 - 4s^3p + 2sp^2,$

multiply (B) by the second, and $\quad x^4y + xy^4 \qquad = s^3p - 3sp^2,$

∴ by subtracting $\quad x^5 + y^5 \qquad = s^5 - 5s^3p + 5sp^2.$ (D)

The general formula is

$$x^n + y^n = s^n - ns^{n-2}p + n \cdot \frac{n-3}{2} s^{n-4}p^2 - n \cdot \frac{n-4}{2} \cdot \frac{n-5}{3} s^{n-6}p^3 + n \cdot \frac{n-5}{2} \cdot \frac{n-6}{3} \cdot \frac{n-7}{4} s^{n-8}p^4 - \text{\&c.}$$

Ex. 6. Given $x + y = a$

$x^3 + y^3 = b$, to find the values of x and y.

Put $\quad xy = p.$

Then (B) Ex. 4. $\quad x^3 + y^3 \quad = a^3 - 3ap,$

∴ $\quad a^3 - 3ap = b,$

and $\quad p = \dfrac{a^3 - b}{3a},$

∴ $\quad xy = \dfrac{a^3 - b}{3a}$

But from the first $\quad x^2 + xy = ax,$

∴ by subtraction $\quad x^2 = ax - \dfrac{a^3 - b}{3a},$

and by transposition $x^2 - ax = -\dfrac{a^3 - b}{3a}.$

Completing the square $4x^2 - 4ax + a^2 = \dfrac{4b - a^3}{3a}$,

and $$2x - a = \pm\sqrt{\left(\frac{4b-a^3}{3a}\right)},$$

∴ $$x = \frac{a}{2} \pm \frac{a}{2}\sqrt{\left(\frac{4b-a^3}{3a}\right)},$$

and $$y = a - x = \frac{a}{2} \mp \frac{a}{2}\sqrt{\left(\frac{4b-a^3}{3a}\right)}.$$

Ex. 7, Given $x - y = 2a$
$x^5 - y^5 = b^5$, to find the values of x and y.

Put $x = z + a$ and $y = z - a$,
then $x^5 = z^5 + 5az^4 + 10a^2z^3 + 10a^3z^2 + 5a^4z + a^5$,
and $y^5 = z^5 - 5az^4 + 10a^2z^3 - 10a^3z^2 + 5a^4z - a^5$,
∴ $x^5 - y^5 = 10az^4 + 20a^3z^2 + 2a^5 = b^5$.

Transposing and completing the square, we have
$$z^4 + 2a^2z^2 + a^4 = \frac{b^5 + 8a^5}{10a},$$

∴ $$z^2 + a^2 = \pm\sqrt{\left(\frac{b^5+8a^5}{10a}\right)},$$

and $$z = \pm\sqrt{\left\{-a^2 \pm \sqrt{\left(\frac{b^5+8a^5}{10a}\right)}\right\}}.$$

Consequently $$x = a \pm \sqrt{\left\{-a^2 \pm \sqrt{\left(\frac{b^5+8a^5}{10a}\right)}\right\}},$$

and $$y = -a \pm \sqrt{\left\{-a^2 \pm \sqrt{\left(\frac{b^5+8a^5}{10a}\right)}\right\}}.$$

Second Solution. Divide the second equation by the first, and we have
$$x^4 + x^3y + x^2y^2 + xy^3 + y^4 = \frac{b^5}{2a}.$$

But $(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 = 16a^4$,

∴ by subtraction $5x^3y - 5x^2y^2 + 5xy^3 = \dfrac{b^5 - 32a^5}{2a}$.

Multiply the square of the first by $5xy$, and we have
$$5x^3y - 10x^2y^2 + 5xy^3 = 20a^2xy,$$

by subtraction $$5x^2y^2 = \frac{b^5 - 32a^5}{2a} - 20a^2xy,$$

$$\therefore \quad 5x^2y^2 + 20a^2xy = \frac{b^5 - 32a^5}{2a},$$

or $$x^2y^2 + 4a^2xy = \frac{b^5 - 32a^5}{10a}.$$

Completing the square $$x^2y^2 + 4a^2xy + 4a^4 = \frac{b^5 + 8a^5}{10a},$$

$$\therefore \quad xy + 2a^2 = \pm\sqrt{\left(\frac{b^5 + 8a^5}{10a}\right)},$$

consequently $$xy = -2a^2 \pm \sqrt{\left(\frac{b^5 + 8a^5}{10a}\right)},$$

but from the 1st $x^2 - xy = 2ax$,

$$\therefore \quad x^2 - 2ax = -2a^2 \pm \sqrt{\left(\frac{b^5 + 8a^5}{10a}\right)},$$

and $$x = a \pm \sqrt{\left\{-a^2 \pm \sqrt{\left(\frac{b^5 + 8a^5}{10a}\right)}\right\}},$$

and $$y = -a \pm \sqrt{\left\{-a^2 \pm \sqrt{\left(\frac{b^5 + 8a^5}{10a}\right)}\right\}}.$$

Ex. 8. Given $\left.\begin{aligned} x^2 + xy &= 70 \\ xy - y^2 &= 12 \end{aligned}\right\}$ to find the values of x and y.

Assume $x = vy$, then $x^2 = v^2y^2$ and $xy = vy^2$.

Consequently we have from the first equation $v^2y^2 + vy^2 = 70$,

and from the second $vy^2 - y^2 = 12$. (A)

Dividing the first by the second,

we have $$\frac{v^2 + v}{v - 1} = \frac{35}{6},$$

$$\therefore \quad 6v^2 + 6v = 35v - 35.$$

Consequently $$6v^2 - 29v = -35,$$

and $$v = \frac{7}{3} \text{ or } \frac{5}{2}.$$

Substituting the first value in (A), it becomes

$$\frac{7}{3}y^2 - y^2 = 12,$$

or $$7y^2 - 3y^2 = 36,$$
whence $$y = \pm 3,$$
and $$x = vy = \pm 7.$$

If the second value of v is substituted in (A), it becomes

$$\frac{5}{2}y^2 - y^2 = 12,$$

whence $$3y^2 = 24,$$
and $$y = \pm 2\sqrt{2},$$
$\therefore$ $$x = vy = \pm 5\sqrt{2}.$$

These equations might be solved by finding the value of x in the last equation, and substituting in the first. Thus,

From the second $$x = \frac{12}{y} + y,$$

$\therefore$ $$x^2 + xy = \frac{144}{y^2} + 24 + y^2 + 12 + y^2 = 70.$$

Clearing of fractions and transposing

$$2y^4 - 34y^2 = -144,$$

whence $$y = \pm 3 \text{ or } \pm 2\sqrt{2},$$

and $$x = \frac{12}{y} + y$$
$$= \pm 7 \text{ or } \pm 5\sqrt{2}.$$

Ex. 9. Let $\left.\begin{aligned} x + y + \sqrt{(x+y)} &= 12 \\ x^3 + y^3 &= 189 \end{aligned}\right\}$ to find the values of x and y.

Put $$x + y = s,$$
then from the first $$s + \sqrt{s} = 12,$$
whence $$4s + 4\sqrt{s} + 1 = 49,$$
and $$\sqrt{s} = 3 \text{ or } -4,$$
$\therefore$ $$x + y = s = 9 \text{ or } 16,$$
consequently $$x^3 + 3x^2y + 3xy^2 + y^3 = 729 \text{ or } 4096.$$
But $$x^3 \qquad\qquad + y^3 = \quad 189.$$
By subtraction $$3x^2y + 3xy^2 = 540 \text{ or } 3907.$$
But $$3x^2y + 3xy^2 = 3xy(x+y) = 27xy \text{ or } 48xy,$$
consequently $$27yx = 540,$$
or $$48xy = 3907,$$
and $$xy = 20 \text{ or } \frac{3907}{48} = 81\frac{19}{48}.$$

But $$x+y=9 \text{ or } 16,$$

whence $$x=5 \text{ or } 4;\ \text{or } 8\pm\frac{1}{4}\sqrt{-\frac{835}{3}},$$

and $$y=4 \text{ or } 5 \text{ or } 8\mp\frac{1}{4}\sqrt{-\frac{835}{3}}.$$

Ex. 10. Given $\left.\begin{aligned}x+xy+xy^2+xy^3=15\\ x^2+x^2y^2+x^2y^4+x^2y^6=85\end{aligned}\right\}$ to find the values of x and y.

Multiplying the equations respectively by $1-y$, and $1-y^2$ they become

$$x-xy^4=15\,(1-y),$$

and $$x^2-x^2y^8=85\,(1-y^2),$$

$\therefore$ by dividing the second by the first, we have

$$x+xy^4=\frac{17}{3}\,(1+y).$$

Dividing this into the first equation, we have

$$\frac{1+y+y^2+y^3}{1+y^4}=\frac{45}{17\,(1+y)}.$$

Clearing of fractions and transposing, this becomes

$$28\,y^4-34\,y^3-34\,y^2-34\,y+28=0. \qquad \text{(A)}$$

Assume $$y^2+1=my,$$

then $$28\,y^4+56y^2+28=28\,m^2y^2,$$

and $$-34\,y^3-34\,y=-34\,my^2,$$

$\therefore$ by addition

$$28\,y^4-34\,y^3+56\,y^2-34\,y+28=28\,m^2y^2-34\,my^2.$$

Subtract the equation (A) from this,

and $$90\,y^2=28\,m^2y^2-34\,my^2,$$

or $$14\,m^2-17\,m=45,$$

whence $$m=\frac{5}{2} \text{ or } -\frac{9}{7},$$

consequently $$y^2+1=\frac{5}{2}y \text{ or } -\frac{9}{7}y,$$

and $$y=2 \text{ or } \frac{1}{2} \text{ or } \frac{-9\pm\sqrt{-115}}{14},$$

$\therefore$ $$x=15\,\frac{y-1}{y^4-1}=1 \text{ or } 8 \text{ or } \frac{(23\mp\sqrt{-115})\,7^3}{9315\mp153\sqrt{-115}}.$$

Equation (A) in the above solution belongs to a class of equations which will be treated of more fully hereafter. (See *Recurring Equations.*) It is sufficient to remark here, that such equations, which are characterized by the coefficients *recurring* in regular order, may always be reduced by a substitution like the above, or by its equivalent,

$$n = y + \frac{1}{y}.$$

Ex. 11. Given $\left.\begin{array}{r} xy^5 + x = 99 \\ xy^4 + xy^3 + xy^2 + xy = 90 \end{array}\right\}$ to find the values of x and y.

Here, if we divide the second equation by the first, we have

$$\frac{y^4 + y^3 + y^2 + y}{y^5 + 1} = \frac{10}{11},$$

or reducing $\quad \dfrac{y^3 + y}{y^4 - y^3 + y^2 - y + 1} = \dfrac{10}{11}.$

Whence, clearing of fractions and transposing,

$$10\,y^4 - 21\,y^3 + 10\,y^2 - 21\,y + 10 = 0. \qquad \text{(A)}$$

Assume $\quad y^2 + 1 = my$

and $\quad 10\,y^4 + 20\,y^2 + 10 = 10\,m^2y^2,$

also $\quad -21\,y^3 - 21\,y = -21\,my^2,$

$\therefore$ by addition $10\,y^4 - 21\,y^3 + 20\,y^2 - 21\,y + 10 = 10\,m^2y^2 - 21\,my^2.$

Subtract (A) from this,

and $\quad 10\,y^2 = 10\,m^2y^2 - 21\,my^2,$

whence $\quad 10\,m^2 - 21\,m = 10,$

$\therefore \quad m = \dfrac{5}{2},$

but $\quad y^2 + 1 = my = \dfrac{5}{2}y,$

consequently $\quad y = 2 \text{ or } \dfrac{1}{2},$

and $\quad x = \dfrac{99}{y^5 + 1} = 3 \text{ or } 96.$

The other value of m leads to imaginary results, and is therefore omitted.

Ex. 12. Given $\frac{x+\sqrt{x+y}}{x-\sqrt{x+y}}+\frac{x-\sqrt{x+y}}{x+\sqrt{x+y}}=2\frac{9}{40}$
and $y^2-\sqrt{xy^2}=\frac{4x}{9}$,
to find the values of x and y.

Considering x as a known quantity in the second equation, we have, by completing the square,

$$y^2-\sqrt{xy^2}+\frac{x}{4}=\frac{25x}{36},$$

whence $$y-\frac{1}{2}\sqrt{x}=\pm\frac{5}{6}\sqrt{x},$$

$\therefore$ $$y=\frac{4}{3}\sqrt{x} \text{ or } -\frac{1}{3}\sqrt{x}.$$

If in the first equation we make the first fraction equal to z, we have

$$z+\frac{1}{z}=\frac{89}{40},$$

whence $$40z^2-89z=-40,$$

and $$z=\frac{8}{5} \text{ or } \frac{5}{8}.$$

Consequently $$\frac{x+\sqrt{x+y}}{x-\sqrt{x+y}}=\frac{8}{5} \text{ or } \frac{5}{8}.$$

Substituting in this the first value of y, viz., $\frac{4}{3}\sqrt{x}$, we have

$$\frac{x+\frac{7}{3}\sqrt{x}}{x+\frac{1}{3}\sqrt{x}}=\frac{\sqrt{x}+\frac{7}{3}}{\sqrt{x}+\frac{1}{3}}=\frac{5}{8} \text{ or } \frac{8}{5},$$

whence $$\sqrt{x}=3 \text{ or } -\frac{17}{3},$$

and $$x = 9 \text{ or } \frac{289}{9},$$

$\therefore$ $$y = 4 \text{ or } -\frac{68}{9}.$$

Similarly from the second value of y, viz. $-\frac{1}{3}\sqrt{x}$, are obtained the values,

$$\sqrt{x} = \frac{14}{3} \text{ or } -4,$$

whence $$x = \frac{196}{9} \text{ or } 16,$$

and $$y = -\frac{14}{9} \text{ or } \frac{4}{3}.$$

Ex. 13. Given $\left.\begin{aligned} x + y &= s \\ xy &= p \end{aligned}\right\}$ to find the values of x and y.

Ans. $x = \dfrac{s \pm \sqrt{s^2 - 4p}}{2}$ and $y = \dfrac{s \mp \sqrt{5 - 4p}}{2}$.

Ex. 14. Given $\left.\begin{aligned} xy &= 15 \\ x^3 - y^3 &= 98 \end{aligned}\right\}$ to find the values of x and y.

Ans. $x = 5$ or -3,
$y = 3$ or -5.

Ex. 15. Given $\left.\begin{aligned} x^2y^2 + xy &= 1260 \\ x + y &= 12 \end{aligned}\right\}$ to find the values of x and y.

Ans. $x = 7$ or 5 or $6 \pm 6\sqrt{2}$,
$y = 5$ or 7 or $6 \mp 6\sqrt{2}$.

Ex. 16. Given $\left.\begin{aligned} 4x^2 - 2xy &= 12 \\ 2y^2 + 3xy &= 8 \end{aligned}\right\}$ to find the values of x and y.

Ans. $x = \pm 2$ or $\pm \frac{3}{7}\sqrt{7}$,
$y = \pm 1$ or $\mp \frac{8}{7}\sqrt{7}$.

Ex. 17. Given $\left.\begin{aligned} x + y &= 10 \\ x^3 + y^3 &= 370 \end{aligned}\right\}$ to find the values of x and y.

Ans. $x = 7$ or 3, $y = 3$ or 7.

Ex. 18. Given $\left.\begin{array}{r} x+y+\sqrt{x+y}=12 \\ x^3+y^3=189 \end{array}\right\}$ to find the values of x and y.

$$\textit{Ans.}\ x=5 \text{ or } 4 \text{ or } 8\pm\frac{1}{4}\sqrt{-\frac{835}{3}},$$
$$y=4 \text{ or } 5 \text{ or } 8\mp\frac{1}{4}\sqrt{-\frac{835}{3}}.$$

Ex. 19. Given $\left.\begin{array}{r} x+y=7 \\ x^5+y^5=1267 \end{array}\right\}$ to find the values of x and y.

$$\textit{Ans.}\ x=4 \text{ or } 3 \text{ or } \frac{7\pm3\sqrt{-11}}{2},$$
$$y=3 \text{ or } 4 \text{ or } \frac{7\mp3\sqrt{-11}}{2}.$$

Ex. 20. Given $\left.\begin{array}{r} \sqrt{y}+\sqrt{x}:\sqrt{y}-\sqrt{x}::\sqrt{x}+2:1 \\ \dfrac{\sqrt{y}+2}{\sqrt{x}}-1=\dfrac{3\sqrt{x}+1+\sqrt{\frac{y}{x}}}{\sqrt{y}} \end{array}\right\}$ to find the values of x and y.

$$\textit{Ans.}\ x=1 \text{ or } \frac{1}{9},\ y=4 \text{ or } \frac{16}{9}.$$

Ex. 21. Given $\left.\begin{array}{r} x+y+z=25 \\ xy=6 \\ yz=60 \end{array}\right\}$ to find the values of x, y, and z.

$$\textit{Ans.}\ x=2 \text{ or } \frac{3}{11},\ y=3 \text{ or } 22, \text{ and } z=20 \text{ or } \frac{30}{11}.$$

Ex. 22. Given $\left.\begin{array}{r} x^3y+x^2y^2=252 \\ x^5y^3=2187 \end{array}\right\}$ to find the values of x and y.

$$\textit{Ans.}\ x=\pm9 \text{ or } \pm9\sqrt{-1} \text{ or } \sqrt[4]{\pm\frac{1}{3}} \text{ or } \pm\sqrt{-\frac{1}{3}\sqrt{3}},$$
$$y=\pm\frac{1}{3} \text{ or } \pm\frac{1}{3}\sqrt{-1} \text{ or } \pm27\sqrt[4]{\frac{1}{3}} \text{ or } \pm27\sqrt{-\frac{1}{3}\sqrt{3}}.$$

Ex. 23. Given $\left.\begin{matrix} y^4 - 432 = 12\,xy^2 \\ y^2 = 12 + 2\,xy \end{matrix}\right\}$ to find the values of x and y. *Ans.* $x = 2$, $y = 6$.

Ex. 24. Given $\left.\begin{matrix} xy + x^2y = \frac{21}{32} \\ x^2 + x + y = \frac{29}{16} \end{matrix}\right\}$ to find the values of x and y.

$$\textit{Ans.} \begin{cases} x = \frac{3}{4} \text{ or } -\frac{7}{4}, & y = \frac{1}{2} \\ x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}, & y = \frac{21}{16}. \end{cases}$$

Note.—In the following examples but one result is given; the student, however, should always obtain all the answers.

Ex. 25. Given $\left.\begin{matrix} x^4y^2 + x^2y^4 = 2900 \\ x^8y^4 + x^4y^8 = 6410000 \end{matrix}\right\}$ to find the values of x and y. *Ans.* $x = 5$, $y = 2$.

Ex. 26. Given $\left.\begin{matrix} x^2 + y^2 - x - y = 249740 \\ xy + x + y = 8516 \end{matrix}\right\}$ to find the values of x and y. *Ans.* $x = 500$, $y = 16$.

Ex. 27. Given $\left.\begin{matrix} 9\frac{x^2}{y^2} + 36\frac{x}{y} = 85 \\ \frac{3\,xy^3}{5} + \frac{51}{10}xy^2 = \frac{102}{5}x + 16 \end{matrix}\right\}$ to find the values of x and y. *Ans.* $x = 3\frac{1}{3}$, $y = 2$.

Ex. 28. Given $\left.\begin{matrix} 7\,x - 3\,y = 29 \\ 2\,x^2 + 4\,y^2 = 66 \end{matrix}\right\}$ to find the values of x and y. *Ans.* $x = 5$, $y = 2$.

Ex. 29. Given $\left.\begin{matrix} x^2y + xy^2 = 120 \\ x^3y^2 + x^2y^3 = 1800 \end{matrix}\right\}$ to find the values of x and y. *Ans.* $x = 5$, $y = 3$.

Ex. 30. Given $\left.\begin{matrix} x + y + xy = 34 \\ x^2 + y^2 = 52 \end{matrix}\right\}$ to find the values of x and y. *Ans.* $x = 6$, $y = 4$.

Ex. 31. Given $\left.\begin{aligned}(x-y)(x^2-y^2)&=288\\(x+y)(x^2+y^2)&=3060\end{aligned}\right\}$ to find the values of x and y. *Ans.* $x=11$, $y=7$.

Ex. 32. Given $\dfrac{x+y+\sqrt{x^2-y^2}}{x+y-\sqrt{x^2-y^2}}=\dfrac{9(x+y)}{8y}$

and $(x^2+y)^2+x-y=2x(x^2+y)+506$

to find the values of x and y. *Ans.* $x=5$, $y=3$.

133. *Questions producing Quadratic Equations.*

The following questions, though given under *Quadratic Equations*, may many of them be solved by simple, or by *pure* equations. The student should endeavour to work them out in as many different ways as possible; it being far more important to acquire the command of analysis which such exercise will give him, than merely to solve any given number of examples.

EXAMPLES.

Ex. 1. A merchant sold a quantity of cloth for \$39, gaining thereby as much per cent. as the cloth cost him. What was the cost of the cloth?

Let $x=$ the price of the cloth,

then $x=$ the gain per cent.

$\therefore \dfrac{x}{100}\cdot x=\dfrac{x^2}{100}=$ the gain on x dollars.

Consequently $\dfrac{x^2}{100}+x=39$,

and $x^2+100x=3900$.

Whence we readily obtain

$x=30$ or -130.

The last value being excluded by the nature of the problem, the price was 30 dollars.

Ex. 2. A bought linen and muslin for \$10.50, the whole number of yards being 50; and each cost as many cents per yard as there were yards of the other. How much of each did he purchase?

Let $x =$ the number of yards of linen,
then $50 - x =$ the number of muslin,
also $50x - x^2 =$ the cost of the linen in cents,
and $50x - x^2 =$ the cost of the muslin,
$\therefore\ 100x - 2x^2 = 1050,$
and $x^2 - 50x = -525.$
$\therefore\ x = 35$ or 15 the number of yards of linen,
and $50 - x = 15$ or 35 the number of muslin.

Ex. 3. The plate of a looking-glass is 36 inches by 27, and is framed with a frame of equal width all round it, the area of the frame being half that of the glass. What is the width of the frame?

Let $x =$ the width of the frame in inches. Then the length of the glass to the outside of the frame is

$$36 + 2x,$$

and its breadth $27 + 2x.$
$\therefore\ (36+2x)(27+2x) = 972 + 126x + 4x^2 =$ the whole area,
and $36 \times 27 = 972 =$ the area of the glass.
Consequently $4x^2 + 126x = 486 =$ area of the frame,
and $4x + 63 = \pm\sqrt{5913} = \pm 77$ very nearly.
Hence $4x = 14$ or $-140,$
and $x = 3\frac{1}{2}$ the breadth required, the negative result evidently not answering the conditions of the problem.

Ex. 4. A and B hired a pasture, in which A put 4 horses, and B as many as cost him \$4.50 per week. B afterwards put in 2 more horses, and found he must pay \$5 per week. At what rate was the pasture hired?

Let $x =$ the number B put in at first,
then $\frac{450}{x} =$ the cost per head in cents,
and $4 \times \frac{450}{x} = \frac{1800}{x} =$ the sum A paid.
$\therefore\ \frac{1800}{x} + 450 =$ the price of the pasture.

In the second case the number of horses is $x + 6$.

$$\therefore \quad x + 6 : x + 2 :: \frac{1800}{x} + 450 : 500$$

$$:: 1800 + 450\,x : 500\,x,$$

consequently, (Art. 43,)

$$500\,x^2 + 3000\,x = 450\,x^2 + 2700\,x + 3600.$$

Transposing $\quad 50\,x^2 + 300\,x = 3600,$

whence $\quad x^2 + 6\,x = 72,$

and $\quad x = 6 \text{ or } -12,$

$$\therefore \quad \frac{1800}{x} + 450 = 750.$$

Hence \$7.50 was the price of the pasture per week.

Ex. 5. There are four numbers in arithmetical progression, the product of the extremes being 22, and that of the means 40. What are the numbers?

Let $x - 3\,y$, $x - y$, $x + y$, and $x + 3\,y$ represent the numbers, then we will have

$$x^2 - 9\,y^2 = 22$$

$$x^2 - y^2 = 40.$$

Whence x and y are readily found to be $6\frac{1}{2}$ and $1\frac{1}{2}$. The numbers are consequently 2, 5, 8, and 11.

Ex. 6. A starts from Philadelphia, towards Pittsburg, travelling uniformly at the rate of 30 miles per day. After he had been gone $2\frac{1}{2}$ days, B starts in pursuit, travelling 15 miles the first day, 20 the second, and so on in arithmetical progression. In what time will he overtake A?

Let $\quad x =$ the number of days required,

then $\quad x + \frac{5}{2} =$ the number A travelled.

Also $30x + 75 =$ the whole number of miles A travelled,

and, (Art. 59,) $\frac{x}{2}(30 + (x - 1)\,5) = \frac{25\,x}{2} + \frac{5\,x^2}{2} =$ the distance B travelled.

$$\therefore \quad \frac{5\,x^2}{2} + \frac{25\,x}{2} = 30\,x + 75,$$

or transposing, &c., $x^2 - 7x = 30$,
whence $x = 10$ or -3.

The first of these answers the conditions required; as for the other, it would indicate that B overtook A 3 days before they set out, which is manifestly absurd. Had the question been stated as below, both results would have applied.

A and B are travelling the same road. A, proceeding uniformly 30 miles a day, arrives at Philadelphia $2\frac{1}{2}$ days before B. The number of miles B travels each day forms an increasing arithmetical progression, the common difference being 5, and the number of miles B travels the day he leaves Philadelphia being 15. How many days from the time B was at Philadelphia were they together?

This question will give the same equation as the former, and the results, 10 and -3, indicate that they would be together at two points on their route, viz., 3 days before and 10 days after B left Philadelphia.

Ex. 7. It is required to find four numbers in proportion, such that their sum may be 20, the sum of their squares 130, and the sum of their cubes 980.

Let w, x, y, and z represent the numbers. Then we have

$$w + x + y + z = 20$$
$$w^2 + x^2 + y^2 + z^2 = 130$$
$$w^3 + x^3 + y^3 + z^3 = 980.$$

Assume $x + y = s$ and $xy = p$.
Then $w + z = 20 - s$ and $wz = p$. (Art. 45.)
Also (Ex. 5, page 159,)

$$x^2 + y^2 = s^2 - 2p$$
$$w^2 + z^2 = (20 - s)^2 - 2p = 400 - 40s + s^2 - 2p,$$
$$\therefore w^2 + x^2 + y^2 + z^2 = 400 - 40s + 2s^2 - 4p = 130. \text{(A)}$$

Again, (Ex. 5, p. 159,) $x^3 + y^3 = s^3 - 3sp$,

$$w^3 + z^3 = (20 - s)^3 - 3(20 - s)p,$$
$$= 8000 - 1200s + 60s^2 - s^3 - 60p + 3sp,$$
$$\therefore w^3 + x^3 + y^3 + z^3 = 8000 - 1200s + 60s^2 - 60p = 980.$$

From (A) $6000 - 600s + 30s^2 - 60p = 1950,$

$\therefore$ $2000 - 600s + 30s^2 = -970,$

and $s^2 - 20s = -99,$

consequently $s = 11$ or 9,

whence (A) $p = 18$,

or $x+y = 11$ or 9 and $xy = 18$,

and $x = 9$ or 6,

$y = 2$ or 3,

also $w + z = 20 - s = 9$ or 11,

$wz = 18$,

$\therefore$ $w = 6$ or 9,

and $z = 3$ or 2.

Hence the numbers are 6, 9, 2, and 3.

Ex. 8. The sum of five numbers in geometrical progression is 31, and the sum of their squares 341, to determine the numbers.

Let $\frac{x^2}{y}, x, y, z, \frac{z^2}{y}$ be the numbers.

Then $\frac{x^2}{y} + x + y + z + \frac{z^2}{y} = 31$.

and $\frac{x^4}{y^2} + x^2 + y^2 + z^2 + \frac{z^4}{y^2} = 341$,

also $xz = y^2$,

put $x + z = s$,

then $x^2 + z^2 = s^2 - 2y^2$, (Ex. 5, p. 159.)

and $\frac{x^2}{y} + \frac{z^2}{y} = \frac{s^2}{y} - 2y$,

$\therefore$ $\frac{s^2}{y} - y + s = 31$,

or $31y - s^2 + y^2 - sy = 0$. (A)

But, from the first equation $\frac{x^2}{y} + \frac{z^2}{y} = 31 - s - y$,

$\therefore$ $\frac{x^4}{y^2} + 2xz + \frac{z^4}{y^2} = 961 - 62s - 62y + s^2 + 2sy + y^2$,

and $\frac{x^4}{y^2} + \frac{z^4}{y^2} = 961 - 62s - 62y + s^2 + 2sy - y^2$,

but $x^2 + y^2 + z^2 = s^2 - y^2$

∴ by addition,

$$\frac{x^4}{y^2}+x^2+y^2+z^2+\frac{z^4}{y^2}=961-62s-62y+2s^2+2sy-2y^2=341$$

To this equation add twice (A)

and $961-62s=341,$

whence $62s=620,$

and $s=10.$

Substituting this value in (A,) we have by reduction,

$$y^2+21y=100,$$

whence $y=4$ or $-25.$

The latter value will lead to imaginary results.

We have also $x+z=10,$

and $xz=y^2=16,$

whence $x=2,$ and $z=8;$

∴ $\frac{x^2}{y}=1,$ and $\frac{z^2}{y}=16,$

and the numbers are 1, 2, 4, 8 and 16.

The above solution is from Simpson's Algebra, and is remarkable for the beauty of some of the reductions. The following, on the principle of recurring equations, though not shorter, is more direct.

Let x, xy, xy^2, xy^3 and xy^4 represent the numbers,

then $x+xy+xy^2+xy^3+xy^4=31,$

and $x^2+x^2y^2+x^2y^4+x^2y^6+x^2y^8=341,$

divide the second by the first,

and $x-xy+xy^2-xy^3+xy^4=11,$

add this to the first,

and $2x+2xy^2+2xy^4=42,$

also by subtraction $2xy+2xy^3=20,$

∴ $20y^4+20y^2+20=42y^3+42y,$

and by transposition $20y^4-42y^3+20y^2-42y+20=0,$

or $10y^4-21y^3+10y^2-21y+10=0$ (A).

Put $y^2+1=my,$

then $10y^4+20y^2+10=10m^2y^2.$

Sub't (A) from this, and $21y^3+10y^2+21y=10m^2y^2,$

but $21y^3 \qquad +21y=21my^2,$

∴ by subtraction $10y^2=10m^2y^2-21my^2,$

or $$10m^2 - 21m = 10,$$

whence $$m = \frac{5}{2} \text{ or } -\frac{2}{5},$$

consequently $$y^2 + 1 = \frac{5}{2}y,$$

and $$y = 2 \text{ or } \frac{1}{2}.$$

This, substituted in the first equation, will give

$$x = 1 \text{ or } 16,$$

whence the numbers are 1, 2, 4, 8 and 16.

The second value of m will lead to imaginary results.

Ex. 9. It is required to divide a line of 15 yards in length, so that the rectangle of the whole and less part may be equal to the square of the greater.

Ans. The parts are $-\frac{15}{2} + \frac{15}{2}\sqrt{5}$ and $\frac{45}{2} - \frac{15}{2}\sqrt{5}$.

Ex. 10. Bought some cloth for \$24, for which I paid \$2 more per yard than there were yards in length. How many yards were there? *Ans.* 4.

Ex. 11. There are two numbers whose product is to 8 times their sum as 3 is to 5, and the difference of whose squares is 80. What are the numbers? *Ans.* 12 and 8.

Ex. 12. Divide 100 into two such parts, that if each be divided by the other, the sum of the quotients may be $2\frac{36}{91}$.

Ans. The parts are 35 and 65.

Ex. 13. The length of a room exceeds its breadth by 8 feet, and the number of yards required to cover it with matting, four feet wide, exceeds $\frac{4}{5}$ the number of feet in the breadth, by 20. Required the dimensions of the room.

Ans. 28 feet by 20.

Ex. 14. A gentleman has a rectangular yard, 100 feet by 80; and wishes to make a gravel walk of equal width half round it. What must be its breadth in order that it may occupy $\frac{1}{4}$ of the ground? *Ans.* 11·8975 feet.

Ex. 15. The product of two numbers is 156, and their sum added to the sum of their squares is 338; what are the numbers? *Ans.* 12 and 13.

Ex. 16. The fore wheel of a carriage makes 6 revolutions more than the hind wheel in going 120 yards; but if the periphery of each wheel be increased one yard, it will make only 4 revolutions more than the hind wheel in the same distance. Required the circumference of each.

Ans. 4 yards and 5 yards.

Ex. 17. A certain number consisting of three digits in geometrical progression, is to the sum of its digits as 124 to 7: and if 594 be added to it, the digits will be inverted. Required the number. *Ans.* 248.

Ex. 18. A sets out from Philadelphia to travel east, at the rate of 20 miles per day. B starts west at the same time, and travels 1 mile the first day, 4 the second, and so on in arithmetical progression. In how many days will they meet, and how far will each have gone, supposing the parallel of latitude through Philadelphia to be 18921 miles in length?

Ans. 106 days. A's distance =2120 m.; B's =16801 m.

Ex. 19. A sets out from New York towards Washington, and travels 1 mile the first hour, 2 the second, 3 the third, and so on. B starts 5 hours after, and travels uniformly 12 miles per hour. In what time will they be together?

Ans. 3 or 10 hours.

Ex. 20. A and B engage to reap a field for $18·00; and as A could reap it in 9 days, they promise to complete it in 5 days. Finding, however, they were unable to finish it, they called in C to assist them the last 2 days, in consequence of which B received 75 cents less than he otherwise would have done. In what time could B or C alone have reaped the field?

Ans. B in 15 days, and C in 18 days.

Ex. 21. A man bought two cubical stacks of hay for \$123, each of which cost $\frac{3}{16}$ as many dollars per solid yard as there were yards in the side of the other. Now, as the greater stood on 9 square yards more than the other, what was the cost of each? *Ans.* \$75 and \$48.

Ex. 22. There are three numbers in geometrical progression whose sum is 7, and the sum of whose squares is 21. What are the numbers? *Ans.* 1, 2 and 4.

Ex. 23. The sum of two numbers multiplied by the greater is 104, and the sum of their squares 89. What are the numbers? *Ans.* 8 and 5 or $\frac{13}{2}\sqrt{2}$ and $\frac{3}{2}\sqrt{2}$.

Ex. 24. There are three numbers in harmonical proportion whose sum is 191; and the product of the extremes 4032. What are the numbers? *Ans.* 56, 63, and 72.

Ex. 25. What two numbers are there whose sum, product, and difference of their squares are equal?

Ans. $\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$ and $\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$.

Ex. 26. There are two numbers, the sum of the squares of which is 58, and the cube of their sum is to the sum of their cubes as 100 to 37. What are the numbers?

Ans. 7 and 3.

Ex. 27. A starts upon a journey, travelling 7 miles the first day, and increasing his day's journey in arithmetical progression so that at the end of a certain number of days he has travelled 282 miles. Now, had he gone but 3 miles the first day, and increased his day's journey by a number of miles one greater than in the former case, he would have gone 300 miles in the same time. Required the number of days occupied by the journey. *Ans.* 12 days.

Ex. 28. Required to divide a line of 134 yards in length into three such parts that the sum of their squares may be 6036, and that the first, twice the second, and three times the third may together make 278.

Ans. The parts are 40, 44, and 50.

Ex. 29. There are four numbers in arithmetical progression, such that the product of the extremes is 3250, and of the means 3300. What are the numbers?

Ans. 50, 55, 60, and 65.

Ex. 30. The sum of six numbers in arithmetical progression is 33, and the sum of squares 199. What are the numbers? *Ans.* 3, 4, 5, 6, 7, and 8.

Ex. 31. Bacchus caught Silenus asleep by the side of a full cask, and seized the opportunity of drinking, which he continued for two-thirds of the time Silenus would have required to empty the whole cask. After that, Silenus awoke and drank what Bacchus had left. Had they drunk both together it would have been emptied two hours sooner, and Bacchus would have drunk only half what he left Silenus. Required the time in which each would have emptied the cask separately. *Ans.* Bacchus in 6 hours, and Silenus in 3.

Ex. 32. There are three numbers in arithmetical progression, such that the square of the first, added to the product of the other two, is 16, and the square of the second, added to the product of the other two, is 14. What are the numbers? *Ans.* 1, 3, and 5.

Ex. 33. A man being asked how many years he had been employed where he then was, replied that the first year he had occupied the post he had received $500, and that his salary had been increased $75 every year. Notwithstanding his expenses each year had absorbed the interest on his former earnings, and half his salary, he had laid by $8500. How many years had he been employed?

Ans. 16 years.

Ex. 34. The arithmetic mean between two numbers exceeds the harmonic mean by 25, and the geometric by 13. What are the numbers? *Ans.* 104 and 234.

Ex. 35. The sum of two numbers is 8, and the sum of their fifth powers is 3368. What are the numbers?

Ans. 3 and 5.

Ex. 36. What number is that, which being increased by 12 and the sum divided by $\frac{1}{4}$ the product of the digits, the quotient may be equal to $2\frac{1}{3}$ times the difference of the digits; and if 27 be added to the number, its digits will be inverted? *Ans.* 58.

Ex. 37. If the values of gold and silver are as 13 to 1, what is the proportion of the two metals in each of two mixtures, such that the value of an ounce of the first may be to that of an ounce of the second as 11 to 17; but if the quantity of gold in each mixture be doubled, then the value of one ounce of the first would be to that of one ounce of the second as 7 to 11?

Ans. The proportion of gold to silver in the first mixture is 1 to 9, and in the second 1 to 4.

Ex. 38. The sum of four numbers in arithmetical progression is 20, and the sum of their reciprocals $\frac{25}{24}$. What are the numbers? *Ans.* 2, 4, 6, and 8.

Ex. 39. There are five whole numbers, the first three of which are in geometrical progression, and the last three in arithmetical progression, the common difference being the second number. The sum of the last four is 40, and the product of the second and fifth is 64. Required the numbers. *Ans.* 2, 4, 8, 12, and 16.

Ex. 40. There are four numbers in arithmetical progression, which being increased by 2, 4, 8, and 15, respectively, the results will be in geometrical progression. Required the numbers. *Ans.* 6, 8, 10, and 12.

Ex. 41. There are three numbers, the difference of whose differences is 3; their sum is 21; and the sum of the squares of the greatest and least is 137. Required the numbers. *Ans.* 4, 6, and 11.

Ex. 42. The sum of four numbers in geometrical progression is 30, and the sum of their squares 340. What are the numbers? *Ans.* 2, 4, 8, and 16.

Ex. 43. The sum of five numbers in geometrical progression is 242, and the sum of their squares 29524. Required the numbers. *Ans.* 2, 6, 18, 54, and 162.

Ex. 44. The sum of the first and last of six numbers in geometrical progression is 488, and the sum of the four means is 240. What are the numbers?

Ans. 2, 6, 18, 54, 162, and 486.

CHAPTER VII.

ON THE PROPERTIES AND SOLUTION OF EQUATIONS.

SECTION I.

On the Fundamental Properties of Equations.

134. Any value of the unknown quantity that satisfies the conditions expressed by the equation is called a root of that equation. We have already seen that quadratic equations have two roots; we shall hereafter show that every equation has as many roots as there are units in the number of its degree.

135. If a be a root of the equation,

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots . Px + R = 0,$$

then will the left member of this equation be divisible by $(x - a)$. For if said division leave a remainder s, we will have (Q representing the quotient)

$$x^n + Ax^{n-1} + \text{ \&c.}, = (x - a)\,Q + s = 0.$$

But $\quad x - a = 0 \;\therefore\; s = 0,$

Cor. The converse of this proposition is evidently true. For if $x - a$ be a factor of the equation,

$$V = x^n + Ax^{n-1} \ldots . Px + R = 0,$$

the quotient being Q, we have

$$V = (x - a)\,Q = 0,$$

which may be satisfied by making $x - a = 0$, that is $x = a$.

136. Every equation has as many roots as there are units in the index of the highest power of the unknown quantity.

Let a be a root of the equation

$$x^n + Ax^{n-1} \ldots . Px + R = 0. \qquad \text{(A)}$$

Then by the last article this expression is equal to

$$(x - a)\,(x^{n-1} + A_1x^{n-2} \ldots + N_1x + P_1) = 0,$$

Now this equation may be satisfied either by making

$$x - a = 0, \text{ or } x^{n-1} + A_1x^{n-2} \dots N_1x + P_1 = 0,$$

the former of which gives $x = a$.

If b be a root of the second equation, this may be written

$$(x - b)(x^{n-2} + A_2x^{n-3} \dots N_2) = 0.$$

Pursuing this investigation, the original equation will be decomposed into the factors $x - a$, $x - b$, &c. So that we will have

$$x^n + Ax^{n-1} + Bx^{n-2} \dots Px + R = (x - a)(x - b)(x - c) \dots (x - p) = 0.$$

But this last equation is satisfied by making

$$x = a, b, c \dots \text{or } p.$$

Consequently these are roots of the equation (A).

This demonstration assumes that every equation has at least one root.

Cor. 1. It must not be understood that these roots are necessarily all different. In fact any number of them, as a, b, c, may be equal. In such cases these are still considered separate roots. If two of the roots are equal to a, the equation will evidently be divisible by $(x - a)^2$. If there be three roots equal to a, by $(x - a)^3$; and so on for any number.

Cor. 2. Since the equation

$$x^n + Ax^{n-1} \dots Px + R = 0 \text{ is equivalent to}$$

$$(x - a)(x - b)(x - c) \dots (x - p) = 0,$$

we must have, (Art. 81,)

A = the sum of the roots with their signs changed.
B = the sum of the products taken two and two.
C = the sum of the products taken three and three.
:
P " . . . . $(n - 1)$ and $(n - 1)$.
R = the product of the roots.

The signs being understood as being changed in every case.

137. No equation has more roots than there are units in the index of the highest power of the unknown quantity.

$$\text{Let } V = x^n + Ax^{n-1} \ldots . Px + R = 0$$

be an equation whose roots are $a, b, c \ldots . p$, we will have

$$V = (x - a)(x - b)(x - c) \ldots . (x - p) = 0.$$

Now, if possible, let a' be a root differing from either of these, then we will have, by substituting this value in the above,

$$V = (a' - a)(a' - b)(a' - c) \ldots . (a' - p) = 0.$$

But this equation is impossible, since none of the factors of V is equal to 0.

138. If an equation has integral coefficients, that of the highest power being unity, the roots cannot be rational fractions.

Let, if possible, $x = \frac{a}{b}$, a and b being prime to each other, be a root of the equation,

$V = x^n + Ax^{n-1} + Bx^{n-2}$ &c., $= 0$. A, B, &c., being integers: then we will have, substituting $\frac{a}{b}$ for x,

$$\frac{a^n}{b^n} + A\frac{a^{n-1}}{b^{n-1}} + B\frac{a^{n-2}}{b^{n-2}} + \ldots . R = 0.$$

Multiplying both members by b^{n-1}, the equation becomes

$$\frac{a^n}{b} + Aa^{n-1} + Ba^{n-2}b + \ldots . Rb^{n-1} = 0.$$

Now the first term, $\frac{a^n}{b}$, is a fraction, while the remaining terms are integral, consequently the function cannot equal 0, and, therefore, $\frac{a}{b}$ is not a root.

Cor. It follows from the above, that the roots of an integral equation, of which the first coefficient is unity, are either integers, surds, or imaginary quantities.

NOTE.—We shall, in what follows, consider the first coefficient unity, and the others integral. The propositions will in this way lose none of their generality, as we shall find hereafter that any equation, whose coefficients are fractional, may be transformed into another of the kind required.

139. If the signs of the alternate terms of an equation be changed, the signs of all the roots will be changed.

Let a be a root of the equation

$$x^n + Ax^{n-1} + Bx^{n-2} \ldots . Px + R = 0. \qquad (1)$$

Then will $-a$ be a root of the equation

$$x^n - Ax^{n-1} + Bx^{n-2} \ldots . \pm Px \mp R = 0. \qquad (2)$$

$$\text{or } -x^n + Ax^{n-1} - Bx^{n-1} \ldots . \mp Px \pm R = 0. \qquad (3)$$

These last equations are evidently identical.

If now we substitute a in equation (1), and $-a$ in (2) and (3), the results will be

$$a^n + Aa^{n-1} + Ba^{n-2} \ldots . + Pa + R, \text{ in } (1)$$

and either $a^n + Aa^{n-1} + Ba^{n-2} \ldots . + Pa + R$,

or $\quad -a^n - Aa^{n-1} - Ba^{n-2} \ldots . - Pa - R$, in each of the others. But these expressions being identical with the first, are each equal to zero, and therefore $-a$ is a root of the equations (2) and (3).

140. Surds and imaginary roots enter an equation by pairs. So that if $a + \sqrt{b}$ be one root, $a - \sqrt{b}$ will be another.

For, if $(a + \sqrt{b})$ be substituted for x in the equation,

$$V = x^n + Ax^{n-1} + Bx^{n-2} \ldots . Px + R = 0,$$

it will become

$$V = (a + \sqrt{b})^n + A(a + \sqrt{b})^{n-1} + \ldots \&c. = 0. \qquad (A)$$

If, now, we expand the powers of the binomial in this equation, it will evidently consist of two parts, one rational, and one composed of surds. So that we will have

$$V = S + U\sqrt{b} = 0. \qquad (B)$$

Now, this cannot be equal to zero, unless we have separately

$$S = 0, \text{ and } U\sqrt{b} = 0,$$

and consequently $\quad S - U\sqrt{b} = 0.$

If we examine the structure of equation (A), we will readily perceive that the irrational part, $U\sqrt{b}$, of equation (B) arises from the odd powers of $\sqrt{b}$ in the developments of the binomials, and must therefore change its sign with $\sqrt{b}$, this being, moreover, all the change that will be produced by substituting $-\sqrt{b}$ for $+\sqrt{b}$.

The substitution of $a - \sqrt{b}$ for x in the equation

$$V = 0,$$

will therefore give

$$S - U\sqrt{b},$$

which we have shown to be equal to zero.

Consequently $a - \sqrt{b}$ is a root.

The same demonstration will apply to the case of imaginary roots. These are of the form

$$a \pm b\sqrt{-1}.$$

141. From what has been said in last article, it is obvious that if $a + b\sqrt{-1}$, be one imaginary root, $a - b\sqrt{-1}$ must be another. The equation will therefore be divisible by $x - a - b\sqrt{-1}$, and also by $x - a + b\sqrt{-1}$. Consequently, their product

$$x^2 - 2ax + a^2 + b^2,$$

must be a quadratic divisor of the equation, and this factor is necessarily positive, whatever value we give to x, for it is evidently equal to $(x - a)^2 + b^2$, the sum of two squares.

Cor. 1. The roots of an equation of an even degree may be all impossible; but if they are not all impossible, two at least must be real.

Cor. 2. Since the quadratic factors containing the corresponding pairs of impossible roots are essentially positive, it is clear that when all the roots are impossible, the product of the quadratic factors is essentially positive, and therefore the absolute number R must be positive. (Art. 136, Cor. 2.)

Cor. 3. Every equation of an odd degree has at least one rational root of a contrary sign to that of the last term; and every equation of an even degree, the last term of which is negative, has at least two real roots, with contrary signs.

142. An equation cannot have a greater number of positive roots than there are variations of signs, in successive terms, nor can it have a greater number of negative roots than there are continuations of the same sign from one term to the next.

Let $+ + - + - + + +$ be the order of signs in any equation. If, then, we introduce a new positive root a, we must multiply the equation by $x - a = 0$. The signs in the operation will be as follows:

$$\begin{array}{l} + + - + - + + + \\ \underline{\quad - - + - + - - -} \\ + \pm - + - + \pm \pm -, \end{array}$$

in which it is apparent that each permanency is changed into an ambiguity by the introduction of the new root, so that the number of continuations of the same sign cannot be increased by the introduction of a positive root, and the number of signs being increased by unity, there must be at least one more variation. Hence the introduction of a positive root increases the number of variations, by one at least. Now, since in the binomial equation $x - a = 0$ we have one variation and one positive root, it follows from what has been said above, that the number of positive roots can never exceed the number of variations of sign.

If we change the signs of the alternate terms in the above, the continuations will become variations, and the variations, permanencies. But, by this change of sign, the signs of all the roots are changed. (Art. 139.) Hence, since this equation cannot have a greater number of positive roots than there are variations of signs; it follows that the original equation cannot have a greater number of negative roots than there are continuations of sign.

Cor. 1. If all the roots are real, the number of positive roots will be *equal* to the number of variations, and that of negative roots equal to the number of permanencies of sign.

Cor. 2. This rule, which is due to Descartes, will sometimes enable us to determine whether there are impossible roots in an equation. For example, suppose it were desirable to know the nature of the roots of the cubic equation,

$$x^3 + Ax + N = 0.$$

Supplying the second term so as to make the equation complete, it becomes

$$x^3 \pm 0x^2 + Ax + N = 0.$$

Now, if we take the upper sign, there are three permanencies, and, consequently, there are no positive roots.

But, if we take the lower sign, there are two variations, and therefore can be but one negative root. The other two must then be imaginary.

SECTION II.

Transformation of Equations.

143. To transform an equation into another whose roots shall be equal to those of the original equation increased or diminished by a given number.

Let $x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} \ldots Px + R = 0$ (1) be an equation. If in this we substitute $y - r$ for x, the resulting equation will evidently have its roots equal to those of the equation increased by r. But as this operation is very tedious, especially for equations of a high degree, we shall point out the following shorter method of arriving at the same result.

By the substitution above proposed we will arrive at an equation

$$y^n + A'y^{n-1} + B'y^{n-2} + C'y^{n-3} \ldots P'y + R' = 0. \qquad (2)$$

If in this we put $(x + r)$ instead of y, it becomes

$$(x + r)^n + A'(x + r)^{n-1} + \ldots\ldots P'(x + r) + R' = 0. \quad (3)$$

Now, this equation must be identical with (1), since (2) was obtained by substituting $y - r$ for x in (1), and (3) by substituting $x + r$ for y in (2), which is nothing more than reversing the operation.

Since, then, (3) and (1) are identical, if each be divided by $(x + r)$, the quotients and remainders must be the same. But (3) divided by $(x + r)$ gives for quotient $(x+r)^{n-1} + A'(x + r)^{n-2} \ldots P'$, and for remainder R'. If this quotient be again divided by $x + r$, the remainder will be P'. This operation may be continued, dividing each quotient by $x + r$, and the several remainders will be the coefficients of the various terms, beginning with the last.

Thus let it be required to find an equation whose roots shall exceed by 4 the roots of the equation

$$x^4 - 5x^3 + 12x^2 + 7x - 12 = 0,$$

dividing by $x+4$, the operation will be as below.

$$
\begin{array}{l}
x+4)\,x^4 - 5x^3 + 12x^2 + 7x - 12\,(x^3 - 9x^2 + 48x - 185 \\
\quad x^4 + 4x^3 \\
\quad\quad -9x^3 + 12x^2 \\
\quad\quad -9x^3 - 36x^2 \\
\quad\quad\quad\quad 48x^2 + 7x \\
\quad\quad\quad\quad 48x^2 + 192x \\
\quad\quad\quad\quad\quad -185x - 12 \\
\quad\quad\quad\quad\quad -185x - 740 \\
\quad\quad\quad\quad\quad\quad\quad 728, \text{ 1st remainder.}
\end{array}
$$

$$
\begin{array}{l}
x+4)\,x^3 - 9x^2 + 48x - 185\,(x^2 - 13x + 100 \\
\quad x^3 + 4x^2 \\
\quad\quad -13x^2 + 48x \\
\quad\quad -13x^2 - 52x \\
\quad\quad\quad\quad 100x - 185 \\
\quad\quad\quad\quad 100x + 400 \\
\quad\quad\quad\quad\quad -585, \text{ second rem.}
\end{array}
$$

$$
\begin{array}{l|l}
x+4)\,x^2 - 13x + 100\,(x - 17 & x+4)\,x - 17\,(1 \\
\quad x^2 + 4x & \quad x + 4 \\
\quad -17x + 100 & \quad\quad -21, \text{ 4th rem.} \\
\quad -17x - 68 & \\
\quad\quad\quad 168, \text{ 3d rem.} &
\end{array}
$$

Hence the transformed equation is

$$x^4 - 21x^3 + 168x^2 - 585x + 728 = 0.$$

By using synthetic division, this operation will be much shortened. Thus:

$$
\begin{array}{lr|l}
 & & 1 \quad -5 + 12 \quad +7 \quad -12 \\
 & -4 & \quad\;\; -4 + 36 - 192 \;\; +740 \\
\hline
\text{1st quotient,} & & 1 \;\; -9 \;\; +48 - 185, \; +728, \text{ 1st rem.} \\
 & & \quad\;\; -4 \;\; +52 - 400 \\
\hline
\text{2d quotient} & & 1 - 13 + 100, - 585, \text{ 2d remainder.} \\
 & & 1 \;\; -4 \;\; +68 \\
\hline
\text{3d quotient,} & & 1 - 17, \quad 168, \text{ 3d remainder.} \\
 & & \quad -4 \\
\hline
 & & 1, -21, \text{ 4th remainder.}
\end{array}
$$

And the equation is as above,

$$x^4 - 21\,x^3 + 168\,x^2 - 585\,x + 728 = 0.$$

The operation may be still better arranged thus, placing the -4 to the right, in the place occupied in ordinary division by the quotient, and omitting all the numbers in the left hand column except the first. Thus

1	−5	12	7	−12 (−4
	−4	36	−192	740
	−9	48	−185	728 1st rem.
	−4	52	−400	
	−13	100	−585, 2d remainder.	
	−4	68		
	−17	168, 3d remainder.		
	−4			
	−21, 4th remainder.			

To transform the equation $3\,x^4 - 4\,x^3 + 7\,x^2 + 8\,x - 12 = 0$ into another whose roots shall be less by 3 than those of the given equation.

3	−4	7	8	−12 (3
	9	15	66	222
	5	22	74	210, 1st rem.
	9	42	192	
	14	64	266, 2d remainder.	
	9	69		
	23	133, 3d remainder.		
	9			
	32, 4th remainder.			

and the equation is

$$3\,x^4 + 32\,x^3 + 133\,x^2 + 266\,x + 210 = 0.$$

Examples.

Ex. 1. Diminish the roots of the equation

$$4\,x^3 - 32\,x^2 - x + 8 = 0$$

by 4.8.

The operation will stand thus,

4	-32	-1	$+8$	(4.8
	16	-64	-260	
	-16	-65	-252	
	16	0		
	0	-65		
	16			
	16	-65	-252	
	3.2	15.36	-39.712	
	19.2	-49.64	-291.712	
	3.2	17.92		
	22.4	-31.72		
	3.2			
	25.6			

Hence the transformed equation is

$$4x^3 + 25.6x^2 - 31.72x - 291.712 = 0,$$

or

$$x^3 + 6.4x^2 - 7.93x - 72.928 = 0.$$

Ex. 2. Transform the equation

$$x^3 - 10x^2 + 27x - 18 = 0$$

into one whose roots shall be less by 3 than those of the given equation. *Ans.* $x^3 - x^2 - 6x = 0.$

Ex. 3. Diminish the roots of

$$x^4 - 8x^3 + 14x^2 + 4x - 8 = 0$$

by 5. *Ans.*

Ex. 4. Increase the roots of

$$x^3 + 5x^2 + 7x + 29 = 0$$

by 7.3. *Ans.*

Ex. 5. Diminish the roots of

$$2x^3 + 3x^2 - 4x - 10 = 0$$

by 7. *Ans.*

Ex. 6. Diminish the roots of

$$x^4 - 8x^3 + 16x^2 + 7x - 12 = 0$$

by 2. *Ans.* $x^4 - 8x^2 + 7x + 18 = 0.$

144. The solution of the last example of the preceding article makes known the method of transforming an equation so as to eliminate the second term.

It is only necessary in the equation

$$x^n + Ax^{n-1} + \&c., = 0$$

to increase or diminish the roots by $\frac{A}{n}$, according as the coefficient of the second term is positive or negative.

EXAMPLES.

Deprive the following equations of their second terms.

Ex. 1. $x^3 - 6x^2 + 8x - 9 = 0.$

Ans.

Ex. 2. $x^4 + 10x^3 - 4x^2 + 8x - 11 = 0.$

Ans.

Ex. 3. $x^3 - 9x^2 + 26x - 34 = 0.$

Ans.

Ex. 4. $x^4 + 8x^3 - 6x - 17 = 0.$

Ans.

Ex. 5. $x^3 - 5x^2 + 8x - 12 = 0.$

Ans. $x^3 - \frac{16}{3}x - 16\frac{7}{27} = 0.$

145. The removal of the second term of a quadratic equation leads at once to the general formula for its solution.

Let $$x^2 + Ax + B = 0$$
be any quadratic equation.

If we transform it so that the roots may become $x' + r$, the result will be

$$x'^2 + (A + 2r)x' + r^2 + Ar + B = 0,$$

and that the second term may vanish, we must have

$$A + 2r = 0, \text{ or } r = -\frac{A}{2},$$

whence $$r^2 + Ar + B = -\frac{1}{4}A^2 + B,$$

the equation therefore becomes

$$x'^2 - \frac{1}{4}A^2 + B = 0,$$

whence $$x' = \pm\sqrt{\frac{1}{4}A^2 - B},$$

and $$x = x' + r = -\frac{A}{2} \pm \sqrt{\frac{1}{4}A^2 - B},$$

which are the values resulting from the ordinary mode of solution.

146. To transform an equation into another, whose roots shall be the reciprocals of those of the given equation.

This is done by substituting $\frac{1}{x}$ for x in the given equation and clearing of fractions. The result will evidently have the same coefficients in an inverted order.

Thus, if $Ax^n + Bx^{n-1} + Cx^{n-2} \ldots Px^2 + Qx + R = 0$ be an equation, the reciprocal equation will be

$$Rx^n + Qx^{n-1} + Px^{n-2} \ldots Cx^2 + Bx + A = 0.$$

Cor. Hence we may transform an equation into another, whose roots shall be greater or less than the reciprocals of the given equation, by applying the process pointed out in Art. 143, to the coefficients taken in a reverse order.

For example, let it be required to transform the equation

$$3x^4 - 13x^3 + 7x^2 - 8x - 9 = 0$$

into one, whose roots shall be equal to the reciprocals of those of the given equation, increased by 2.

−9	−8	7	−13	3	(−2
	18	−20	26	−26	
	10	−13	13	−23	
	18	−56	138		
	28	−69	151		
	18	−92			
	46	−161			
	18				
	64				

and the transformed equation is

$$-9x^4 + 64x^3 - 161x^2 + 151x - 23 = 0.$$

147. If the coefficients of the proposed equation be the same when taken in an inverted order, it is evident that the equation, whose roots are the reciprocals of the roots of the given equation, will be identical with it, and will therefore furnish the same series of roots. The roots of the original equation must therefore be of the form

$$a, \frac{1}{a}; \ b, \frac{1}{b}; \ c, \frac{1}{c}, \ \&c.$$

If the equation be of an odd degree, and the coefficients taken in reverse order be of like magnitudes as when taken in direct order, but with signs all different, then will the roots of the transformed also be identical with those of the original equation, for by changing all the signs of the original, which of course produces no change in the roots, the signs of the corresponding terms will be the same, and the equations will therefore be identical.

The same reasoning will hold with equations of an even degree, provided the middle term be absent.

Such equations are called *recurring* equations.

148. A *recurring* equation of an *odd* degree has one of its roots equal to $+1$, or -1, according as the signs of the like coefficients are *different* or *alike*.

For, since every power of $+1$ is positive, if the signs of the like coefficients be different, the substitution of $+1$ for x will render the corresponding terms equal, and of contrary signs, they will therefore destroy each other; but if the signs of the equal coefficients be the same, then, since one of them will belong to an even power, and the other to an odd one, the substitution of -1 for x will make the corresponding terms equal and of opposite signs.

149. A recurring equation of an even degree, in which the like coefficients have different signs, and whose middle term is wanting, is divisible by x^2-1, and has of course two roots, viz., $+1$ and -1.

For, let $x^{2n}+Ax^{2n-1}+Bx^{2n-2}\ldots-Bx^2-Ax=1-0$ be a recurring equation of the kind required. It may be written

$$x^{2n}-1+Ax(x^{2n-2}-1)+Bx^2(x^{2n-4}-1)+, \&c., =0,$$

the first member of which is evidently divisible by x^2-1.

It is moreover evident that the depressed equation will be a recurring one of the $(2n-2)$th degree:

17

For the resulting equation is

$$\begin{array}{l|l|l|l|l} x^{2n-2}+Ax^{2n-3}+B & x^{2n-4}+A & x^{2n-5}\ldots+A & x^3+B & x^2+Ax+1=0 \\ \qquad\qquad\qquad +1 & \qquad +C & \qquad\quad +C & \quad +1 & \\ & \qquad \&c. & \qquad\quad \&c. & & \end{array}$$

which is a recurring equation of an even degree, the equal coefficients having like signs.

150. To transform an equation into another whose roots are some multiple or submultiple of the given equation.

Let $x^n+Ax^{n-1}+Bx^{n-2}\ldots\ldots Px+Q=0$ be an equation.

Put $y=mx$ or $x=\dfrac{y}{m}$, and we have

$$\frac{y^n}{m^n}+A\frac{y^{n-1}}{m^{n-1}}+B\frac{y^{n-2}}{m^{n-2}}\ldots\ldots P\frac{y}{m}+Q=0,$$

clearing of fractions, and

$$y^n+Amy^{n-1}+Bm^2y^{n-2}\ldots\ldots Pm^{n-1}y+Qm^n=0$$

is the required equation.

This equation is evidently formed by multiplying the second coefficient by m, the 3d by m^2, &c.

Cor. 1. If an equation have fractional coefficients; it may be changed into one with integral coefficients, by transforming it so that the roots shall be equal to those of the proposed equation, multiplied by the least common multiple of the de nominators.

Cor. 2. If the successive coefficients of an equation be divisible by m, m^2, &c., then m is a common measure of the roots.

SECTION III.

On the Limits of the Roots of Equations.

151. In the numerical solution of equations it is often of importance to determine the limits between which the real roots must be found. The limits ∞ and 0, are evidently the extreme limits, between which all positive roots must lie; and 0 and $-\infty$ are equally limits to the negative roots. But in order to obtain the numerical value of the roots it is evident we must discover much narrower limits than these. The principal means of obtaining these will be found in the following part of this section.

A superior limit to the positive roots is a number numerically greater than the greatest positive root; and an inferior limit of the negative roots is one numerically greater, abstraction being made of its sign, than the greatest negative root.

A superior limit is characterized by the property, that it or any number greater than it, when substituted for x in the equation, causes the result to be positive; and an inferior limit necessarily produces a negative result, as likewise do all greater negative numbers, provided the equation is of an odd degree.

152. In any equation whose second term is negative, and all the other terms positive, the coefficient of the second term, taken positively, is a superior limit to the roots.

Let the equation be

$$x^n - Ax^{n-1} + Bx^{n-2} + \ldots\ldots + Nx + R = 0.$$

Now it is evident, that substituting A for x renders the first two terms equal; the equation will therefore be reduced to

$$BA^{n-2} + CA^{n-3} \ldots\ldots + NA + R.$$

The result of the substitution is therefore positive, unless A is a root of the equation

$$BA^{n-2} + CA^{n-3} \ldots\ldots + NA + R = 0,$$

which is impossible, because it has no changes of signs, and consequently no positive root. (Art. 142.)

If any number greater than A be substituted for x, the first two terms, $x^n - Ax^{n-1}$, give a positive result, and hence the whole result is also positive. A is therefore a superior limit.

153. The greatest negative coefficient increased by unity is a superior limit.

Let $-$D be the greatest negative coefficient of the equation

$$x^n + Ax^{n-1} + Bx^{n-2} + \ \&c. \ldots + Nx + R = 0. \quad (1)$$

Then D + 1 is a superior limit.

Comparing the equation with the following, viz.,

$$x^n - Dx^{n-1} - Dx^{n-2} \ldots\ldots - Dx - D = 0, \qquad (2)$$

it is evident that any number which is a superior limit to this will likewise be to the other.

Now the latter may be written

$$x^n - D(x^{n-1} + x^{n-2} \ldots\ldots x + 1) = 0,$$

or $x^n - D\left\{\dfrac{x^n - 1}{x - 1}\right\} = 0.$ (Art. 63.)

If, now, $D + 1$, be substituted for x in the first member of this equation, it becomes

$$(D+1)^n - D\left\{\frac{(D+1)^n - 1}{D}\right\} = (D+1)^n - (D+1)^n + 1 = 1,$$

a positive quantity.

But if s, a greater quantity than $D + 1$, be substituted for x, the result

$$s^n - D\left\{\frac{s^n - 1}{s - 1}\right\} > 1,$$

for $\qquad s^n - 1 = (s^n - 1).$

Therefore, inasmuch as $s > D + 1$ or $s - 1 > D$,

$$\frac{s^n - 1}{D} > \frac{s^n - 1}{s - 1},$$

or $$s^n - 1 > D\left\{\frac{s^n - 1}{s - 1}\right\}$$

and $$s^n - D\left\{\frac{s^n - 1}{s - 1}\right\} > 1.$$

D is therefore a superior limit to the equation (2), and therefore to (1) likewise.

154. In any equation of the nth degree, if $-Gx^{n-r}$ be the first negative term, and $-P$ the greatest negative coefficient, then will $P^{\frac{1}{r}} + 1$ be a superior limit.

Conceive G and all the subsequent coefficients to be negative and equal to $-P$, which is evidently the most unfavourable case. Then, if we substitute $P^{\frac{1}{g}}$ for x in the inequality

$$x^n > P(x^{n-g} + x^{n-g-1} \ldots\ldots + 1), \qquad (1)$$

it becomes

$$P^{\frac{n}{g}} > P\{P^{\frac{n-g}{g}} + P^{\frac{n-g-1}{g}} \ldots\ldots + 1). \qquad (2)$$

Now, the last member of this inequality is equal to

$$P^{\frac{n}{g}} + P^{\frac{n-1}{g}} \ldots\ldots + P,$$

which, being greater than $P^{\frac{n}{g}}$, renders (2) impossible.

The second member of the inequality (1) is equal (Art. 63) to

$$P\left\{\frac{x^{n-g+1} - 1}{x - 1}\right\}$$

which becomes, by the substitution of $P^{\frac{1}{g}} + 1$ for x,

$$P \,.\, \left\{\frac{(P^{\frac{1}{g}}+1)^{n-g+1} - 1}{P^{\frac{1}{g}}}\right\}$$

$$= P^{\frac{g-1}{g}}\{(P^{\frac{1}{g}} + 1)^{n-g+1} - 1\}$$

$$= P^{\frac{g-1}{g}}(P^{\frac{1}{g}}+1)^{n-g+1} - P^{\frac{g-1}{g}},$$

$$= \left\{\frac{P^{\frac{1}{g}}}{P^{\frac{1}{g}}+1}\right\}^{g-1} . (P^{\frac{1}{g}} + 1)^n - P^{\frac{g-1}{g}},$$

which is evidently less than $(P^{\frac{1}{g}} + 1)^n$, and therefore $P^{\frac{1}{g}} + 1$ substituted for x satisfies the inequality (9), and as it is plain that any number greater than $P^{\frac{1}{g}} + 1$ will likewise satisfy (1), $P^{\frac{1}{g}} + 1$ is a superior limit.

EXAMPLES.

Ex. 1. Required a superior limit to the roots of the equation,

$$x^4 - 5x^3 + 37x^2 - 3x + 39 = 0.$$

Here $\qquad P = 5,\ g = 1,$

$\therefore \qquad P^{\frac{1}{g}} + 1 = 6$, the limit required.

Ex. 2. $x^5 + 7x^4 - 12x^3 - 49x^2 + 52x - 13 = 0.$

Here $P = 49$ and $g = 2$,

$\therefore$ $P^{\frac{1}{2}} + 1 = 7 + 1 = 8$, the limit required.

Ex. 3. $x^4 + 11x^2 - 25x - 67 = 0.$

Ans. The limit is $67^{\frac{1}{3}} + 1 = 6.$

Ex. 4. $3x^3 - 2x^2 - 11x + 4 = 0.$

Divide by 3, and

$$x^3 - \frac{2}{3}x^2 - \frac{11}{3}x + \frac{4}{3} = 0,$$

and the limit required $\frac{11}{3} + 1 = 5.$

155. To determine the inferior limits to the negative roots, it is only necessary to change the signs of the alternate coefficients, by which the signs of the roots will all be changed (Art. 139); inferior limits to the negative roots thus become superior limits to the positive roots, and may be determined as above.

156. If a, b, c, &c., be the real roots of an equation arranged in the order of their magnitude, so that $a > b$, $b > c$, &c.; and if a series of numbers a_1, b_1, c_1, &c., be taken, such that $a_1 > a$, $a > b_1$, $b_1 > b$, $b > c_1$, $c_1 > c$, &c.; then, if a_1, b_1, c_1, &c., be substituted for x in the equation, the first result will be positive, and the others alternately negative and positive.

The original equation is equivalent to

$$(x - a)(x - b)(x - c) \ldots\ldots = 0.$$

This, by the substitution of a_1 for x, becomes

$$(a_1 - a)(a_1 - b)(a_1 - c),$$

the factors of which are all positive, and hence their product must be positive.

If b_1 be substituted, it becomes

$$(b_1 - a)(b_1 - b)(b_1 - c),$$

the first factor being negative, and all the rest positive, the result is therefore negative.

The substitution of c_1 for x renders two of the factors positive, and of course the product is positive, and so on.

Cor. 1. Hence, if we find two numbers, which, when substituted for the unknown quantity, give results of different signs, we may be certain that there is an odd number of roots contained between them.

Cor. 2. By the substitution of the natural series 0, 1, 2, 3, &c., taken negatively as well as positively, we will generally be enabled to discover the real roots. Sometimes, however, there are two or four or some even number of roots contained between two consecutive terms of the natural series; in such cases their existence will not be indicated by this substitution. If, for instance, one of the roots was $\sqrt{3}$, and another $\sqrt{2}$, these both being contained between 1 and 2, the substitution of these latter numbers would afford no indication of them.

Cor. 3. If the equation be of an even degree, the substitution of a quantity less than the least root will produce a positive result; but if the degree be odd, the result will be negative.

157. To find an equation, whose roots are intermediate between the roots of the equation,

$$V = x^n + Ax^{n-1} + Bx^{n-2} + \ldots . Nx^2 + Px + Q = 0.$$

The roots of such an equation being limits to those of the proposed, it is called the *limiting* equation.

In the equation $V=0$, make $x=y+r$, and we shall have

$$\begin{array}{rl}
x^n = & y^n + nry^{n-1} + n.\frac{n-1}{2}r^2y^{n-2} + \ldots \quad nr^{n-1}y + r^n \\
Ax^{n-1} = & Ay^{n-1} + (n-1)Ary^{n-2} + \ldots (n-1)Ar^{n-2}y + Ar^{n-1} \\
Bx^{n-2} = & By^{n-2} + \ldots (n-2)Br^{n-3}y + Br^{n-2} \\
\vdots & \qquad \vdots \\
Px = & Py \quad + Pr \\
Q = & Q \\
\hline
\therefore V = & y^n + A'y^{n-1} + B'y^{n-2} + \ldots \quad P'y + Q' = 0, (1)
\end{array}$$

in which A′, B′, &c., are put for the sum of the coefficients of the different powers of y; so that

$$P' = nr^{n-1} + (n-1)Ar^{n-2} + (n-2)Br^{n-3} + \ldots + P.$$

If, now, a, b, c, d, &c., be the roots of $V=0$, arranged in the order of their magnitudes, the roots of equation 1 will be $a-r$, $b-r$, $c-r$, $d-r$, &c.

Consequently (Art. 136, Cor. 2,)

$$
\begin{aligned}
P' = \;& (r-b)(r-c)(r-d) \text{ to } (n-1) \text{ factors} \\
& +(r-a)(r-c)(r-d) \text{ to } \quad \text{"} \\
& +(r-a)(r-b)(r-d) \ldots\ldots \quad \text{"} \\
& +(r-a)(r-b)(r-c) \ldots\ldots \quad \text{"} \\
& + \&c.
\end{aligned}
$$

If, now, we make $r=a, b, c$, &c., successively in the above, we shall have the following results, viz.,

$$
\begin{aligned}
P' &= (a-b)(a-c)(a-d)\ldots\ldots = +\cdot+\cdot+ = + \\
P' &= (b-a)(b-c)(b-d)\ldots\ldots = -\cdot+\cdot+ = - \\
P' &= (c-a)(c-b)(c-d)\ldots\ldots = -\cdot-\cdot+ = + \\
P' &= (d-a)(d-b)(d-c)\ldots\ldots = -\cdot-\cdot- = -
\end{aligned}
$$

And since the substitution of a, b, c, &c., for r, give results alternately positive and negative, the roots of $P'=0$ must be contained between a, b, c, &c. (Cor. 1. Art. 156.) Consequently, (writing x for r,)

$$P' = nx^{n-1} + (n-1)Ax^{n-2} + (n-2)Bx^{n-3} \ldots . 2Nx + P = 0$$

is the limiting equation required.

158. If the equation $V=0$, have equal roots; these must also be roots of the equation $P'=0$, and hence the two equations must have a common measure.

Thus if a', b', c', &c., be the roots of $P'=0$; and $a=b$, we shall also have $a=a'$, and the factor $x-a$ will be found in both equations.

If $a=b=c$, we shall also have $a=a'=b'$, and $(x-a)^2$ will be the common measure.

To determine the equal roots of an equation, then, it is only necessary to form the limiting equation and find the common measure of the two polynomials. This common measure must be formed by factors containing the equal roots. If, for instance, there are four roots equal to a and three equal to b; the common measure will be

$$(x-a)^3(x-b)^2.$$

EXAMPLES.

Ex. 1. Determine the equal roots of the equation

$$3x^5 - 10x^3 + 15x + 8 = 0.$$

The limiting equation is

$$15x^4 - 30x^2 + 15.$$

The common measure of these (Art. 35) is

$$x^2 + 2x + 1 = (x+1)^2.$$

Hence there are three roots equal to -1.

Ex. 2. Find the roots of the equation

$$x^4 - 14x^3 + 61x^2 - 84x + 36 = 0.$$

The limiting equation is

$$4x^3 - 42x^2 + 122x - 84 = 0,$$

and their common measure is

$$x^2 - 7x + 6 = (x-6) \cdot (x-1.)$$

Hence the roots are 6, 6, 1, and 1.

Ex. 3. The equation

$$x^3 - 7x^2 + 16x - 12 = 0.$$

has equal roots. What are they? *Ans.* 2, 2.

Ex. 4. What are the equal roots of the equation

$$x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108 = 0?$$

Ans. 3, 3, 3, and 2, 2.

Ex. 5. What are the roots of the equation

$$3x^6 - 8x^5 - 24x^4 + 48x^3 + 29x^2 - 12x + 180 = 0?$$

Ans. 3, 3, -2, -2, and $\frac{1}{3} \pm \frac{1}{3}\sqrt{-14}$.

Ex. 6. Solve the equation

$$x^6 - 2x^5 + 6x^4 - 8x^3 + 12x^2 - 8x + 8 = 0,$$

which has equal roots.

Ans. $x = \pm\sqrt{-2}, \pm\sqrt{-2}$, and $1 \pm \sqrt{-1}$.

Ex. 7. The equation

$$x^5 - 8x^4 + 25x^3 - 38x^2 + 28x - 8 = 0$$

has equal roots. What are they?

Ans. 2, 2, 2, 1, and 1.

SECTION IV.

Imaginary and Real Roots.

159. The determination of the number of imaginary roots in an equation has been considered a problem of great difficulty; and though many methods of solution have been discovered, yet as most of them fail in some cases, the problem could not be considered as entirely solved. By the following beautiful theorem of M. Sturm this difficulty has been completely overcome.

Let $\quad Y = x^n + Ax^{n-1} + \ldots . Nx + P = 0$

be an equation of the nth degree, having no equal roots,

and $\quad Y_1 = 0$, the limiting equation.

If there are any equal roots in the proposed equation, these must first be determined by (Art. 158,) and the equation depressed.

Operate with these as though their common measure were desired, calling the several remainders with their signs changed, Y_2, Y_3, &c., Y_m.

The primitive function Y, and the derived functions Y_1, $Y_2 \ldots Y_m$, will be of decreasing dimensions in x, the final one Y_m being independent of that quantity.

Now to determine the number of real roots between any limits p and q, we have only to substitute these values for x in the primitive and derived functions, noting the number of variations of sign in the results. The difference in the number of variations resulting from the two substitutions will be the number of real roots between those limits. If $+\infty$ and $-\infty$ be used instead of p and q, we will have the whole number of real roots.

DEMONSTRATION.

1st. No two consecutive functions can vanish for the same value of x.

For from the mode in which the functions are derived, we have

$$\begin{aligned} Y &= Q\ Y_1 - Y_2 \\ Y_1 &= Q_1 Y_2 - Y_3 \\ Y_2 &= Q_2 Y_3 - Y_4 \\ &\vdots \\ Y_{m-2} &= Q_{m-2} Y_{m-1} - Y_m. \end{aligned}$$

Now if any two, as Y_2 and Y_3 each $= 0$, we will have $Y_4 = 0$, $Y_5 = 0$, $Y_m = 0$, and Y and Y_1 will have a common measure, but this is impossible, as there are no equal roots.

2d. If one of the functions as Y_3 becomes 0, for a particular value of x, the adjacent functions will have contrary signs for that value.

For we have $\quad Y_2 = Q_2\, Y_3 - Y_4.$

But as $\quad Y_3 = 0, \; Y_2 = -Y_4.$

3d. Let p be greater than the greatest, and q less than the least root (negative roots being considered less than corresponding positive ones) of the equations

$$Y = 0, Y_1 = 0, Y_2 = 0 \ldots Y_{m-1} = 0.$$

If, now, we suppose q gradually to increase until it becomes equal to the least root of the above equations, no change can have taken place in the signs of any of the results. At this point, however, that function to which this root belongs, (say Y_3,) vanishes; and as, in this case, the adjacent functions necessarily have different signs, no change in the number of variations of signs can be produced by this circumstance, and, consequently, every change in the number of variations must arise from the change of sign in the primitive function.

Let us suppose the value of q has changed until it has passed the least root of $Y = 0$, but not arrived at that of $Y_1 = 0$, this being necessarily greater than the least of $Y = 0$, (Art. 157.) Now Y and Y_1 produce results of opposite signs if any number less than their least root be substituted in them, (Cor. 3, Art. 156.) Hence the change of sign that takes place in Y, by passing the least root of $Y = 0$, must make them of the same sign, and diminish the number of variations by unity.

If we conceive q still to increase until it has passed the least root of $Y_1 = 0$, this function will have changed its sign, and of course Y and Y_1 will have different signs. As q still increases, it will pass the second root of $Y = 0$, by which operation the number of variations will be again diminished by unity.

Now as no change in the number of variations can arise from the vanishing of any of the derived functions, and as the number of variations is diminished by unity, whenever

q increasing passes one of the roots of $Y=0$; it follows that the whole number of real roots will be equal to the diminution in the number of variations, that is produced by q increasing from its original value to an equality with p.

The calculation will be simplified by using $+\infty$, and $-\infty$, for p and q, since it will only be necessary to note the sign of the first term of each function, as in the first case the signs of the results will be the same as those of the first terms; and in the latter the even powers will be the same and the odd powers different from the signs of those terms in the functions.

As an example, let it be required to find the number of real roots in the equation

$$x^3-4x^2-6x+8=0.$$

Here we have

$$Y = x^3-4x^2-6x+8$$

$$Y_1 = 3x^2-8x-6$$

$$3x^2-8x-6)\,3x^3-12x^2-18x+24\,(x$$

$$3x^3-8x^2-6x$$

$$-4x^2-12x+24$$

$$\times \text{ by } \tfrac{3}{4} \qquad -3x^2-9x+18\,(-1$$

$$-3x^2+8x+6$$

$$Y_2 = 17x-12, \qquad -17x+12)\,3x^2-8x-6\,($$

$$51x^2-136x-102\,(-3x$$

$$51x^2-36x$$

$$-100x-102.$$

The final remainder will evidently be negative. Hence

$$Y_3 = +.$$

The substitution of $+\infty$, and $-\infty$ in these will produce the following results:

$$+\infty \text{ gives } + + + +$$

$$-\infty \text{ gives } - + - +.$$

Now as there are no variations in the first line, and three in the second, the roots are all real.

To determine the initial figures of these roots, we will merely have to substitute the numbers 0, 1, 2, and -1, -2, &c., in the functions, and note the signs of the results. Thus,

$x=$				$x=$			
0	gives	$+--+$	2 variat.	0	gives	$+--+$	2 variat.
1	"	$--++$	1	-1	"	$++-+$	2 "
2	"	$--++$	1	-2	"	$-+-+$	3 "
3	"	$--++$	1				
4	"	$-+++$	1				
5	"	$++++$	0				

Hence the roots are between 0 and 1, 4 and 5, and -1, and -2. The initial figures are therefore 0, 4 and -1.

Again, let the number and situation of the roots in

$$x^3+11x^2-102x+181=0.$$

The functions are

$$Y = x^3 + 11x^2 - 102x + 181$$
$$Y_1 = 3x^2 + 22x - 102$$
$$Y_2 = 122x - 393$$
$$Y_3 = +.$$

The substitution of ∞ gives all the signs positive.
of $-\infty$ gives three variations.

Hence there are three real roots.

$x=0$ gives $+--+$ two variations.
$x=1$ " $+--+$
$x=2$ " $+--+$
$x=3$ " $+--+$ two variations.
$x=4$ " $++++$ no variations.

As there are two roots between 3 and 4, we will transform the functions so that their roots shall be diminished by 3, (Art. 143.) The result will be

$$Y = x^3 + 20x^2 - 9x + 1$$
$$Y_1 = 3x^2 + 40x - 9$$
$$Y_2 = 122x - 27$$
$$Y_3 = +.$$

In these

$x=0$ gives $+--+$ two variations.
$x=.1$ " $+--+$
$x=.2$ " $+--+$ two variations.
$x=.3$ " $++++$ no variations.

We thus find that these roots are both contained between 3.2, and 3.3.

Transforming these functions so that the roots may be less by .2 than those last used, they will become

$$Y = x^3 + 20.6\,x^2 - .88\,x + .008$$
$$Y_1 = 3\,x^2 + 41.2\,x - .88$$
$$Y_2 = 122\,x - 2.6$$
$$Y_3 = +.$$

In these

$x = 0$ gives $+\,-\,-\,+$ two variations.
$x = .01$ " $+\,-\,-\,+$ two "
$x = .02$ " $-\,-\,-\,+$ one "
$x = .03$ " $+\,+\,+\,+$ no "

So that the roots are 3.21, and 3.22; and as the sum of the roots is -11, the third is -17.4.

3. Find the number of real roots in the equation

$$x^6 - 2\,x^5 + 6\,x^4 - 8\,x^3 + 12\,x^2 - 8\,x + 8 = 0.$$

The functions are

$$Y = x^6 - 2\,x^5 + 6\,x^4 - 8\,x^3 + 12\,x^2 - 8\,x + 8$$
$$Y_1 = 6\,x^5 - 10\,x^4 + 24\,x^3 - 24\,x^2 + 24\,x - 8$$
$$Y_2 = -13\,x^4 + 24\,x^3 - 60\,x^2 + 48\,x - 68$$
$$Y_3 = x^3 + 4\,x^2 + 2\,x + 8$$
$$Y_4 = -x^2 - 2$$
$$Y_5 = 0.$$

Since $x^2 + 2$ is the common measure, there are equal roots. Now $x^2 + 2 = (x + \sqrt{-2})\,.\,(x - \sqrt{-2}.)$ Consequently there are two pairs of equal roots, equal respectively to $\sqrt{-2}$ and $-\sqrt{-2}$.

$x = \infty$ gives the signs of the various functions.

$+\,+\,-\,+\,-$ three variations.

$x = -\infty$ gives $+\,-\,-\,-\,-$ one variation.

The above would indicate two real roots, whereas all the roots are imaginary. This failure results from the theorem being applied to a case to which it does not belong. The demonstration was based upon the supposition that there were no equal roots, and upon this supposition one of the important steps was founded. In the above equation, however, there are two pair of equal roots, and of course the demonstration does not apply. The immediate reason of the failure in this instance may be found by substituting -4 for x, which renders $Y_3 = 0$, Y_2 and Y_4, both being negative.

Determine the number and situation of the real roots in each of the following equations:

1. $x^4 - 11x^3 + 22x^2 - 71x - 35 = 0.$
2. $2x^4 - 13x^2 - 30x - 25 = 0.$
3. $x^5 - 5x^4 + 3x^2 + 17x - 32 = 0.$
4. $3x^4 + 17x^3 - 11x^2 - 22x - 13 = 0.$
5. $x^3 - 7x^2 + 9x - 11 = 0.$

160. Let it be required to determine the conditions that all the roots of

$$x^3 + px + q = 0$$

may be real.

Here

$$Y = x^3 + px + q$$
$$Y_1 = 3x^2 + p$$
$$Y_2 = -2px - 3q$$
$$Y_3 = -4p^3 - 27q^2.$$

Now in order that the roots may be all real, there must be three permanencies when $+\infty$ is substituted for x in these functions, and three variations when $-\infty$ is substituted. That the first condition may hold the value of Y_3, viz.

$$-4p^3 - 27q^2 \text{ must be positive,}$$

or

$$4p^3 + 27q^2 < 0,$$

and this cannot be unless p is negative.

If, then, p be negative, and Y_3 positive, we will have by the substitution of $-\infty$ for x, the following order of signs, viz.:

$$- + - +,$$

giving three variations, and thus proving the existence of three *real* roots. The condition, then,

$$4p^3 + 27q^2 < 0$$

is essential, and sufficient to indicate that all the roots are real.

CHAPTER VIII.

ON THE NUMERICAL SOLUTION OF EQUATIONS.

SECTION I.

Cardan's Rule for solving Cubic Equations.

161. Let $x^3 + Ax^2 + Bx + C = 0$, be any equation of the third degree. In order to render it more manageable let it be deprived of its second term, (Art. 144,) and let the resulting equation be

$$x^3 + bx + c = 0.$$

Assume $x = y + z$,

Then $x^3 = y^3 + z^3 + 3yz(y + z,)$

or transposing

$$x^3 - 3yzx - (y^3 + z^3) = 0.$$

Consequently $3yz = -b$ and $y^3 + z^3 = -c$.

$\therefore$ from the 1st $z^3 = -\dfrac{b^3}{27y^3}$,

which being substituted in the other, this becomes

$$y^3 - \frac{b^3}{27y^3} = -c,$$

or clearing of fractions $y^6 + cy^3 = \dfrac{b^3}{27}$.

$\therefore$ solving the quadratic $y^3 = -\dfrac{c}{2} + \sqrt{\left(\dfrac{c^2}{4} + \dfrac{b^3}{27}\right)} = A^3$,

and $z^3 = -\dfrac{c}{2} - \sqrt{\left(\dfrac{c^2}{4} + \dfrac{b^3}{27}\right)} = B^3$,

Consequently, as $x = y + z$, we have the following general formula for the roots of an equation of the third degree.

$$x = \sqrt[3]{\left\{-\frac{c}{2} + \sqrt{\left(\frac{c^2}{4} + \frac{b^3}{27}\right)}\right\}} + \sqrt[3]{\left\{-\frac{c}{2} - \sqrt{\left(\frac{c^2}{4} + \frac{b^3}{27}\right)}\right\}}.$$

This is *Cardan's* formula.

The above formula would appear to give but one of the roots. When, however, it is remembered that the values of

y and z are determined by extracting the cube roots of A^3 and B^3, which operation is equivalent to solving the equations

$$y^3 - A^3 = 0; \quad z^3 - B^3 = 0,$$

it will be seen that each of them must have three values, which are readily determined. Thus,

The first of the above equations is equivalent to

$$(y - A)(y^2 + Ay + A^2) = 0.$$

The first of these factors gives the root A, the other solved as a quadratic will give

$$y = \frac{-1 + \sqrt{-3}}{2}A, \text{ and } y = \frac{-1 - \sqrt{-3}}{2}A.$$

Similarly the other values of z are

$$\frac{-1 + \sqrt{-3}}{2}B, \text{ and } \frac{-1 - \sqrt{-3}}{2}B.$$

It might now appear that the three values of y combined with the three values of z would give nine values for x, and that, consequently, an equation of the third degree has 9 roots.

The reasoning, however, is incorrect, for the values of y and z are subjected to the condition that

$$y^3 z^3 = -\frac{b^3}{27}.$$

Six, however, of the combinations alluded to above give imaginary products, and are therefore to be rejected.

The only values of x are

$$A + B$$

$$\frac{-1 + \sqrt{-3}}{2}A + \frac{-1 - \sqrt{-3}}{2}B,$$

$$\frac{-1 - \sqrt{-3}}{2}A + \frac{-1 + \sqrt{-3}}{2}B.$$

162. If $\frac{c^2}{4} + \frac{b^3}{27}$ be negative, that is, if $4b^3 + 27c^2 < 0$, the values of x apparently become imaginary; although we know (Art. 160) that this is the only case in which they are all real; and as no means have yet been discovered for reducing the complicated imaginary forms to real values, Car-

dan's rule fails to give the roots except when two of them are imaginary. The case in which the rule fails is called the irreducible case, and has occupied much of the attention of many distinguished mathematicians.

EXAMPLES.

Let the equation be

$$x^3 - 6x^2 + 3x + 38 = 0.$$

Here it will first be necessary to remove the second term, for which purpose the roots must be diminished by 2. Thus,

$$\begin{array}{rrrrl} 1 & -6 & +3 & +38 & (2 \\ & 2 & -8 & -10 & \\ \hline & -4 & -5 & 28 & \\ & 2 & -4 & & \\ \hline & -2 & -9 & & \\ & 2 & & & \\ \hline & 0 & & & \end{array}$$

and the equation is

$$y^3 - 9y + 28 = 0.$$

Hence $b = -9$ and $c = 28$ $\therefore \frac{c^2}{4} + \frac{b^3}{27} = 196 - 27 = 169$,

and $y = \sqrt[3]{-14 + 13} + \sqrt[3]{-14 - 13}$,

$= -1 - 3 = -4$.

$\therefore x = y + 2 = -2$.

Ex. 2. Solve $x^3 - 6x^2 + 3x - 18 = 0$, by Cardan's rule.
Ans. $x = 6$.

SECTION II.

Recurring Equations.

163. It has been shown (Art. 147, *et seq.*) that a recurring equation of an odd degree has one of its roots $= +1$ or -1, according as the signs of the equal coefficients are different or alike, and that the remaining roots are the one half reciprocals of the other half; that if the equation be of

an even degree, and have the signs of the equal coefficients alike, the same law holds good respecting the roots; and that if the signs of the equal coefficients be different, and the middle term be absent, two of the roots are $+1$ and -1, and of the rest one half are the reciprocals of the other half.

164. On account of these peculiar properties of recurring equations, they may always be reduced to others of lower dimensions; one of an odd degree may at once be depressed to the next inferior degree, by dividing by the factor $x-1$ or $x+1$, for which purpose the method of *synthetic division* is admirably adapted.

Thus, let it be required to remove the factor $x+1$ from the equation

$$5x^5 - 7x^4 + 8x^3 + 8x^2 - 7x + 5 = 0.$$

The operation is

5	-7	$+8$	$+8$	-7	$+5$	$(-1$
	-5	$+12$	-20	12	-5	
	-12	20	-12	$+5$	0	

The resulting equation is therefore

$$5x^4 - 12x^3 + 20x^2 - 12x + 5 = 0,$$

a recurring equation of the fourth degree.

165. If the equation be of an even degree, the middle term being absent, and the equal coefficients affected with opposite signs, the factor x^2-1 may be eliminated, and the equation thus depressed to a degree lower by 2 than the original one.

For an example, let the equation

$$x^6 - 7x^5 + 9x^4 - 9x^2 + 7x - 1 = 0$$

be proposed.

1	1	-7	9	0	-9	7	-1
0		0	0	0	0	0	
1			1	-7	10	-7	1
	1	-7	10	-7	1		

the resulting equation is therefore

$$x^4 - 7x^3 + 10x^2 - 7x + 1 = 0,$$

a recurring eouation of the fourth degree.

166. A recurring equation depressed as in the last two articles, or one whose degree is even, and the equal roots affected with like signs, may be reduced to another of half its degree.

To prove this, let

$$x^{2n} + Ax^{2n-1} + Bx^{2n-2} \ldots\ldots + Bx^2 + Ax + 1$$

be a recurring equation of the kind required.

Dividing by x^n, it may be written,

$$\left(x^n + \frac{1}{x^n}\right) + A\left(x^{n-1} + \frac{1}{x^{n-1}}\right) + B\left(x^{n-2} + \frac{1}{x^{n-2}}\right)$$
$$+ \text{\&c.}, = 0. \qquad \text{(A)}$$

Now, we have shown, (Ex. 5, p. 159)

that if $x + y = s$ and $xy = p$,

$$x^2 + y^2 = s^2 - 2p,$$
$$x^3 + y^3 = s^3 - 3sp,$$
$$x^4 + y^4 = s^4 - 4s^2p + 2p^2,$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots$$
$$x^n + y^n = s^n - ns^{n-2}p + n \cdot \frac{n-3}{2} s^{n-4}p^2 - n \cdot \frac{n-4}{2} \cdot \frac{n-5}{3} s^{n-6} p^3 + \text{\&c.}$$

Let now $x + \frac{1}{x} = z$, then $x \cdot \frac{1}{x} = 1$,

and the above formulas become

$$x + \frac{1}{x} = z$$
$$x^2 + \frac{1}{x^2} = z^2 - 2$$
$$x^3 + \frac{1}{x^3} = z^3 - 3z \qquad \text{(B)}$$
$$x^4 + \frac{1}{x^4} = z^4 - 4z^2 + 2$$
$$\vdots \quad \vdots \quad \vdots$$
$$x^n + \frac{1}{x^n} = z^n - nz^{n-2} + n \cdot \frac{n-3}{2} z^{n-4} - \text{\&c.}$$

Which values substituted in (A) will give an equation of the nth degree in z, and if the roots of this be determined, those of (A) will be found by solving the quadratic

$$x + \frac{1}{x} = z, \quad \text{or } x^2 - zx = -1.$$

The values of the different functions need not be calculated separately, as it is readily seen that any one may be derived by multiplying the last by z, and subtracting the one immediately preceding. Thus,

$$z^4 - 4z^2 + 2 = (z^3 - 3z)\,z - (z^2 - 2).$$

167. Ex. 1. Let the recurring equation

$$4x^6 - 24x^5 + 57x^4 - 73x^3 + 57x^2 - 24x + 4 = 0$$

be given for solution. Dividing by x^3, and arranging as in (A), it becomes

$$4\left(x^3 + \frac{1}{x^3}\right) - 24\left(x^2 + \frac{1}{x^2}\right) + 57\left(x + \frac{1}{x}\right) - 73 = 0,$$

or substituting the functions (B)

$$4(z^3 - 3z) - 24(z^2 - 2) + 57z - 73 = 0,$$

that is $$4z^3 - 24z^2 + 45z - 25 = 0.$$

By trial, we readily find one root of this equation to be 1, and depressing the equation, we have

$$4z^2 - 20z + 25 = 0,$$

for the equation containing the other roots; and this equation being equivalent to $(2z - 5)\,.\,(2z - 5) = 0$,
has two equal roots, viz.,

$$z = \frac{5}{2}.$$

Having thus obtained the values of z, we have from the equation $$x^2 - zx = -1,$$

the following $$x^2 - x = -1 \text{ and } x^2 - \frac{5}{2}x = -1,$$

of which the first gives

$$x = \frac{1}{2} \pm \frac{1}{2}\sqrt{-3},$$

and the second

$$x = 2 \text{ or } \frac{1}{2},$$

the roots of the given equation are therefore

$$2, \frac{1}{2}, 2, \frac{1}{2}, \text{ and } \frac{1}{2} + \frac{1}{2}\sqrt{-3} \text{ and } \frac{1}{2} - \frac{1}{2}\sqrt{-3}.$$

That the last pair are the reciprocals of each other may be shown by obtaining their product, which will be found to be unity.

Ex. 2. Let the equation

$$x^5 - 11\,x^4 + 17\,x^3 + 17\,x^2 - 11\,x + 1 = 0$$

be given.

This equation has necessarily the root $x = -1$, and depressing, we obtain

$$x^4 - 12\,x^3 + 29\,x^2 - 12\,x + 1 = 0;$$

or dividing by x^2,

$$x^2 + \frac{1}{x^2} - 12\left(x + \frac{1}{x}\right) + 29 = 0,$$

which becomes by substituting the values (B),

$$z^2 - 12\,z + 27 = 0,$$

whence

$$z = 9 \text{ or } 3,$$

and from the equation

$$x^2 - zx = -1,$$

which becomes $x^2 - 9\,x = -1$ and $x^2 - 3\,x = -1$,

we obtain $x = \frac{9}{2} \pm \frac{1}{2}\sqrt{77}$, and $x = \frac{3}{2} \pm \frac{1}{2}\sqrt{5}$.

Consequently, the five roots are

$$-1, \frac{9}{2} + \frac{1}{2}\sqrt{77}, \frac{9}{2} - \frac{1}{2}\sqrt{77}, \frac{3}{2} + \frac{1}{2}\sqrt{5} \text{ and } \frac{3}{2} - \frac{1}{2}\sqrt{5}.$$

Ex. 3. Determine the roots of the equation

$$x^5 - \frac{1}{6}x^4 - \frac{43}{6}x^3 + \frac{43}{6}x^2 + \frac{1}{6}x - 1 = 0.$$

Ans. $1, 2, \frac{1}{2}, -3$ and $-\frac{1}{3}$.

Ex. 4. Determine the roots of the equation

$$x^5 + 2\frac{1}{5}x^4 - 17\frac{3}{5}x^3 - 17\frac{3}{5}x^2 + 2\frac{1}{5}x + 1 = 0.$$

Ans. $-1, -5, -\frac{1}{5}, 2 + \sqrt{3}$, and $2 - \sqrt{3}$.

Ex. 5. Depress the equation

$$3x^7 - 4x^6 + 17x^5 - 25x^4 - 25x^3 + 17x^2 - 4x + 3 = 0.$$

Ans. One root is -1, depressed equation $3y^3 - 7y^2 + 15y - 35 = 0$.

Ex. 6. Depress the equation

$$6x^4 - 21x^3 + 15x^2 - 21x + 6 = 0.$$

Ans. $2y^2 - 7y + 1 = 0$.

Ex. 7. What are the roots of

$$x^5 - 7\frac{1}{5}x^4 + 11\frac{2}{5}x^3 + 11\frac{2}{5}x^2 - 7\frac{1}{5}x + 1 = 0.$$

Ans. $-1, 5, \frac{1}{5}, \frac{3}{2} + \frac{1}{2}\sqrt{5}$ and $\frac{3}{2} - \frac{1}{2}\sqrt{5}$.

Ex. 8. What are the roots of

$$20x^5 - 109x^4 + 146x^3 - 146x^2 + 109x - 20 = 0.$$

Ans. $1, 4, \frac{1}{4}, \frac{1}{10} + \frac{3}{10}\sqrt{-11}$ and $\frac{1}{10} - \frac{3}{10}\sqrt{-11}$.

Ex. 9. What are the roots of

$$30y^4 - 91y^3 + 30y^2 - 91y + 30 = 0.$$

Ans. $3, \frac{1}{3}, -\frac{3}{20} + \frac{1}{20}\sqrt{-391}$ and $-\frac{3}{20} - \frac{1}{20}\sqrt{-391}$.

168. There is a class of equations very analogous to recurring equations that likewise admit of being depressed to others of half their degree. They are of the form

$$Ax^{2n} + Bx^{2n-1} + Cx^{2n-2} + Dx^{2n-3} \ldots \pm Dx^3 \mp Cx^2 \pm Bx \mp A = 0,$$

the upper sign being used when n is odd, and the lower when it is an even number. In equations of this kind the roots are of the form $a, -\frac{1}{a}, b, -\frac{1}{b}$, &c.

First, let n be an odd number, as for instance 3, so that the equation may be of the form

$$Ax^6 + Bx^5 + Cx^4 + Dx^3 - Cx^2 + Bx - A = 0.$$

Dividing by x^3, it becomes

$$A\left(x^3 - \frac{1}{x^3}\right) + B\left(x^2 + \frac{1}{x^2}\right) + C\left(x - \frac{1}{x}\right) + D = 0.$$

Assume now $\quad x - \frac{1}{x} = z,$

then $\quad x^2 + \frac{1}{x^2} = z^2 + 2$

$$x^3 - \frac{1}{x^3} = z^3 + 3z.$$

Hence by substitution, we have

$$\begin{array}{r|l} Az^3 + Bz^2 + 3A & z + 2B = 0 \\ +C & +D \end{array}$$

an equation of the third degree.

Next let n be even, or the equation of the form

$$Ax^8 + Bx^7 + Cx^6 + Dx^5 + Ex^4 - Dx^3 + Cx^2 - Bx + A = 0.$$

Dividing by x^4, it becomes

$$A\left(x^4 + \frac{1}{x^4}\right) + B\left(x^3 - \frac{1}{x^3}\right) + C\left(x^2 + \frac{1}{x^2}\right) + D\left(x - \frac{1}{x}\right) + E = 0.$$

Assuming as before $\quad x - \frac{1}{x} = z,$

we have as before $\quad x^2 + \frac{1}{x^2} = z^2 + 2,\ x^3 - \frac{1}{x^3} = z^3 + 3z,$

also $\quad x^4 + \frac{1}{x^4} = z^4 + 4z^2 + 2.$

Substituting these values, we have

$$\left.\begin{array}{r|r|l} Az^4 + Bz^3 + 4A & z^2 + 3B & z + 2A \\ +C & +D & +2C \\ & & +E \end{array}\right\} = 0.$$

Having determined the value of z from the depressed equation, when it is possible so to do, that of x may be obtained from the equation

$$x - \frac{1}{x} = z, \text{ or } x^2 - zx = 1.$$

EXAMPLES.

Ex. 1. Let it be proposed to solve the equation

$$9x^4 - 3x^3 - 74x^2 + 3x + 9 = 0.$$

Dividing by x^2, and arranging, this becomes

$$9\left(x^2 + \frac{1}{x^2}\right) - 3\left(x - \frac{1}{x}\right) - 74 = 0.$$

This, by the substitution above indicated, becomes

$$9z^2 - 3z = 56.$$

Whence $$z = \frac{8}{3} \text{ or } -\frac{7}{3},$$

and the equations $x^2 - \frac{8}{3}x = 1$, and $x^2 + \frac{7}{3}x = 1$, give the following values for x, viz.,

$$x = 3, -\frac{1}{3}, -\frac{7}{6} + \frac{1}{6}\sqrt{85} \text{ and } -\frac{7}{6} - \frac{1}{6}\sqrt{85}.$$

Ex. 2. Depress the equation

$$x^4 + 6x^3 - 20x^2 - 6x + 1 = 0.$$

Ans. $y^2 + 6y - 18 = 0.$

Ex. 3. What are the roots of

$$4x^4 - 17x^3 - 4x^2 + 17x + 4 = 0.$$

Ans. $2 \pm \sqrt{5}$, and $\frac{1}{8} \pm \frac{1}{8}\sqrt{65}$.

SECTION III.

Determination of Integral Roots by the Method of Divisors.

169. It has been demonstrated (Art. 138) that no equation, in which the coefficient of the first term is unity and the other coefficients integers, can have a fractional root. In such cases the roots must either be integers or interminable decimals. It will be shown in the next section how we may approximate as near as we choose to the true value of those decimal roots, which method will likewise apply to the determination of the integral roots figure by figure. The

following neat method of determining the integral roots was proposed by Newton, and is called the *Method of Divisors*.

Let $x^n+Ax^{n-1}\ldots Fx^5+Gx^4+Hx^3+Lx^2+Nx+P=0$, be an equation of the nth degree, the coefficients being all integers.

Let a be an integral root, then

$$a^n+Aa^{n-1}\ldots Fa^5+Ga^4+Ha^3+La^2+Na+P=0.$$

$$\therefore \frac{P}{a}=-a^{n-1}-Aa^{n-2}\ldots-Fa^4-Ga^3-Ha^2-La-N.$$

Hence every integral root must be a divisor of the last term P. Call $\frac{P}{a}=Q$, and we have by transposition, and dividing by a,

$$\frac{Q+N}{a}=-a^{n-2}-Aa^{n-3}\ldots\ldots-Fa^3-Ga^2-Ha-L.$$

Consequently $\frac{Q+N}{a}$ is an integer. Calling it R, and transposing L and dividing a, we have

$$\frac{R+L}{a}=-a^{n-3}-Aa^{n-4}\ldots\ldots-Fa^2-Ga-H,$$

$\frac{R+L}{a}$ is therefore a whole number.

Proceeding in this manner, we shall evidently obtain $\frac{P}{a}=Q$, $\frac{Q+N}{a}=R$, $\frac{R+L}{a}=S$, $\frac{S+L}{a}=T$, &c., all integers, the last quotient being -1.

170. From the above it appears that if a is an integral root the last coefficient must be divisible by it, so must the sum of the quotient and preceding coefficient; of this quotient and the preceding coefficient, and so throughout, the last quotient being -1.

Having, then, determined the integral divisors of the absolute term of the equation, we must submit all of those between the limits of the roots found by the methods pointed out in chap. 6, sec. 3, to the preceding tests; those which satisfy them all will be roots of the equation.

171. The preceding proposition has been explained above on the principles pointed out by Newton. The following method is perhaps more direct, and, moreover, will serve to point out the formula for calculation.

Let, as before,

$$x^n + Ax^{n-1} \dots + Gx^4 + Hx^3 + Lx^2 + Nx + P = 0,$$

be the equation, of which a is an integral root. The equation is divisible by $(x - a)$, and of course by $a - x$. Writing the coefficients in an inverted order and applying the method of *Synthetic Division*, the operation will stand thus, calling the quotients as before, Q, R, S, &c.

a	P	$+$N	$+$L	$+$H	$+$G	$+$A	$+1$
$+1$		Q	R	S		A$-a$	-1
	P	N$+$Q	L$+$R	H$+$S		$-a$	0
Quotients	Q	R	S	T		-1	

In which it is at once perceived that the sum of each quotient and the next coefficient must be divisible by A, and that the last quotient must be -1.

Having found one of the roots, we may use the depressed equation

$$Q + Rx + Sx^2 \dots\dots - x^{n-1} = 0,$$

or its equivalent

$$P + (N + Q)x + (L + R)x^2 \dots\dots - ax^{n-1} = 0,$$

to determine the subsequent roots.

EXAMPLES.

Let it be required to determine the integral roots of the equation

$$x^5 + 5x^4 + x^3 - 16x^2 - 20x - 16 = 0. \qquad (1)$$

There being but one change of signs, there can be but one positive root.

Now, the superior limit of the roots is (Art. 154)

$$1 + \sqrt[3]{16} \text{ or } 4,$$

and if we change the signs of the alternate terms, the equation becomes

$$x^5 - 5x^4 + x^3 + 16x^2 - 20x + 16 = 0,$$

in which it is readily seen that 5 is greater than the greatest root; -5 is therefore an inferior limit to the roots of (1). The only divisors of 16 between these limits are

$$+2,\ +1,\ -1,\ -2,\ -4,$$

all the others may be rejected. It is also at once seen that $+1$ and -1, will not satisfy the equation; it is only necessary, therefore, to try the remaining divisors.

2)	−16	−20	−16	+ 1	+5	+1	
		− 8	−14	−15	−7	−1	
−2)	−16	−28	−30	−14	−2	0	coeff'ts of 1st depressed equation.
		+ 8	+10	10	2		
−4)	−16	−20	−20	− 4	0,		coeff'ts of 2d equation.
		4	4	+ 4			
	−16	−16	−16	0,			coeff'ts of 3d equation.

Hence 2, -2, and -4, are roots of the equation (1), and the depressed equation is

$$16 + 16x + 16x^2 = 0,$$

or

$$x^2 + x + 1 = 0,$$

of which the roots are imaginary.

Ex. 2. Determine the integral roots of

$$x^4 + x^3 - 62x^2 - 80x + 1200 = 0.$$

Here $\quad 1 + \sqrt{80} = 9$ is a superior limit,

and $\quad -63$ is an inferior limit.

The only factors of 1200 that can be roots, are therefore

8, 6, 5, 4, 3, 2, 1, —1 —2 —3 —4 —5 —6 —8 —12, &c.

Arranging the coefficients and trying the various divisors, beginning with 2, since we can at once see that 1 is not a root, the operation will stand,

2)	1200	−80	−62	1	1
		600	260	99	50
		520	198	100	51

and 2 is not a root.

3)	1200	−80	−62	1	1
		40			
		−40			

40 not being divisible by 3, 3 is not a root. Proceeding in the same manner we shall find 4 is not a root.

5)	1200	−80	−62	1	1
		240	32	−6	−1
6)		160	−30	−5	0
		200	60	5	
		360	30.	0	

Hence 5 and 6 are the positive roots. The depressed equation

$$30\,x^2 + 360\,x + 1200 = 0,$$

or

$$x^2 + 12\,x + 40 = 0,$$

will furnish the imaginary roots,

$$-6 \pm 2\sqrt{-1}.$$

Ex. 3. Determine the roots of

$$x^4 - 5\,x^3 - 5\,x^2 + 45\,x - 36 = 0.$$

Ex. 4. Determine the roots of

$$x^5 - 10\,x^4 + 29\,x^3 - 10\,x^2 - 62\,x + 60 = 0.$$

Ex. 5. Determine the roots of

$$6\,x^4 - 43\,x^3 + 107\,x^2 - 108\,x + 36 = 0.$$

SECTION IV.

Horner's Method for Approximation to the Value of the Roots of an Equation.

172. The discovery of the best method of approximating to the true value of the roots of an equation, has been an object of much attention to mathematicians. Various expedients have been proposed for the purpose, several of which have acquired much celebrity. Amongst these, the method below, which was first published in 1819, by W. G. Horner, of Bath, England, is by far the best, not only on account of its simplicity, but also of its brevity.

The principles upon which it is based are the following.

Let m be a number which differs but little from the root of the equation

$$V = x^n + Ax^{n-1} + \ldots\ldots Nx + P = 0,$$

so that, if $x = m + r$, r may be a small quantity.

Then, if the equation be transformed into another,

$$V' = r^n + A'r^{n-1} \ldots\ldots\ldots N'r + P' = 0,$$

whose roots are less by m than those of the equation

$$V = 0.$$

We may use the last two terms $N'r + P' = 0$, as a trial equation by which to find a near approximation to the value of r, which call m'.

If we again transform the equation

$$V' = 0$$

into one whose roots are equal to those of $V' = 0$ diminished by m'; we may use the last two terms to find m'', a near approximation to the value of m'.

Thus proceeding as far as we wish, and we will have

$$x = m + m' + m'', \&c.$$

173. On the above principles is founded the following

RULE

For approximating to the true value of the roots of an Equation.

1st. Find by Sturm's theorem, or by trial, the situation and first figure of the real roots.

2d. Transform the equation (Art. 143) so that its roots shall be those of the original equation diminished by the part of the root thus discovered.

3d. With the absolute term in this transformed equation for a dividend, and the coefficient of x for a divisor, obtain the next figure of the root.

4th. Again transform the equation so that its roots shall be diminished by the value of the figure last determined, we may thus find another figure, and so proceed until the root has been obtained to as great a degree of accuracy as is desired.

NOTE 1. It sometimes occurs that the sign of the absolute term will change in the course of the operation. Unless this change is accompanied by a change of sign in the coefficient of x, the figure which gives rise to this change must be incorrect.

NOTE 2. To determine the negative roots, change the signs of the alternate terms in the equation (Art. 139), and proceed as before.

Ex. 1. What is the root of the equation

$$x^3 - 17x^2 + 54x - 350 = 0?$$

The integral part of the root is found by trial or by Sturm's theorem to be 14. The subsequent operation will be as follows, viz.:

1	—17	54	—350	(14.954
	14	—42	168	
	— 3	12	—182	
	14	154	170.379	
	11	166	—11.621│	
	14	23.31	10.740│875	
	25.9	189.31	—.880│125	
	9	24.12	.865│275664	
	26.8	213.4│3	—14│849336	
	9	1.3│875		
	2│7.75	214.8│175		
*	5	1.3│900		
	27.80	216.│2075		
	5	.│111416		
	27.854	216.│318916		
	4	.│111432		
	27.858	216.│430348		
	4			
	27.862			

The above operation has been carried on precisely according to the directions of the rule. It will, however, be perceived that more decimals have been used than were necessary to give the root true to the third decimal place. In fact, had all the figures to the right of the vertical lines, and those in the left hand column below the asterisk, been omitted, the result would have been the same, and the labour much abridged.

The operation would then stand thus:

1	−17	54	−350	(14.95407
	14	−42	168	
	−3	12	−182	
	14	154	170.379	
	11	166	−11.621	
	14	23.31	10.741	
	25.9	189.31	−.880	
	9	24.12	865	
	26.8	213.4,3	−15	
	9	1.4	15	
	2,7.7	214.8		
		1.4		
		2,1,6.2		

The above contraction is performed by cutting off from the coefficient of x, after the operation with the figure 9 is completed, one figure, viz. 3, to the right, and from the corresponding coefficient of x^2, the two figures 77, then proceed with the rest as before, until the operation with the figure 5 is completed, after which cut off one figure in the column of x, two in that of x^2, three in that of x^3, and so on.

The following example will still further illustrate the rule.

Ex. 2. Extract the root of the equation

$$x^5 + 4x^4 - 3x^3 + 10x^2 - 2x - 962 = 0.$$

Here the first figure of one of the roots is 3.

1	4	—3	10	—2	—962 (3.385777
	3	21	54	192	570
	7	18	64	190	—392
	3	30	144	624	290.21133
	10	48	208	814	—101.78867
	3	39	261	153.3711	94.64260
	13	87	469	967.3711	— 7.14607
	3	48	42.237	166.6714	6.18160
	16	135	511.237	1133.942,5	— .96447
	3	5.79	44.001	49.090	.86793
	19.3	140.79	555.238	1183.032	— 9654
	.3	5.88	45.792	50.096	8682
	19.6	146.67	601.0,30	1233.12,8	— 972
	3	5.97	12.6	3.19	868
	19.9	152.64	613.6	1236.32	— 102
	3	6,06	12.6	3.19	
	20.2	(15,8.70	626.2	1239.5,1	
	3		12.6	.4	
	20.5		63,8.8	1239.9	
				.4	
				124,0.3	

The root has thus been found true to six places of decimals. If another period had been used, it would have been obtained to 11 decimal places.

Ex. 3. Extract the root of the equation

$$x^4 - 12x^2 + 12x - 3 = 0.$$

Ans. $x = 2.858083$.

Ex. 4. Find a root of the equation

$$x^5 + 2x^4 + 3x^3 + 4x^2 + 5x - 54321 = 0.$$

Ans. $x = 8.4144547$.

Ex. 5. Find the roots of the equation

$$x^3 - 23x - 24 = 0.$$

Ans. 5.250785, — 1.101601 and — 4.149184.

Ex. 6. Required the roots of the equation

$$x^4 - 12x^2 + 12x - 3 = 0.$$

Ans. 2.85808, .60602, .44328, and -3.90738.

Ex. 7. Find all the roots of the equation

$$2x^3 + 3x^2 - 4x - 10 = 0.$$

Ans. 1.624819, the others are imaginary.

Ex. 8. Extract the cube root of 3 to 7 decimals, that is, find the root of

$$x^3 - 3 = 0.$$

Ans. 1.4422496.

Ex. 9. Extract the fifth root of 7.624 to 7 decimals.

Ans. 1.5011932.

SECTION V.

Binomial Equations.

174. Binomial Equations are such as consist of but two terms; the one being the power of some unknown quantity, and the other an absolute number. The most general form under which such equations can be presented is

$$y^n \pm a^n = 0,$$

which, by the substitution of x for $\frac{y}{a}$, is reduced to

$$x^n \pm 1 = 0.$$

It is in this form that they are treated of in this section.

Cor. Since $x = \frac{y}{a}$, $y = ax = x\sqrt[n]{a^n}$.

175. If n be even, the equation $x^n - 1 = 0$ or $x^n = 1$, has two real roots, viz. $+1$ and -1. The binomial $x^n - 1$, is therefore divisible by $(x+1)(x-1) = x^2 - 1$. Performing the division, the equation is reduced to

$$x^{n-2} + x^{n-4} + x^{n-6} + \dots\dots x^2 + 1 = 0,$$

all the roots of which must be imaginary.

With respect to the equation $x^n + 1 = 0$, or $x^n = -1$, all the roots are imaginary, two of them being $\pm\sqrt{-1}$. The expression $x^n + 1$, is therefore divisible by $(x + \sqrt{-1})(x - \sqrt{-1}) = x^2 + 1$; the division leading to the equation

$$x^{n-2} - x^{n-4} + x^{n-6} - \ldots\ldots \pm x^4 \mp x^2 \pm 1 = 0.$$

If n be odd, the equation $x^n + 1 = 0$ or $x^n = -1$ has one real root, viz. -1; the remaining $n - 1$ roots being imaginary.

Similarly, the real root of the equation $x^n - 1 = 0$ is $x = 1$, (n being odd,) the others being imaginary.

176. No two roots of a binomial equation can be equal.

For $$x^n \pm 1 = 0$$

being such an equation, its limiting equation (Art. 157) is

$$nx^{n-1} = 0,$$

which evidently has no root that belongs to the original equation. (See Art. 158.)

177. If a be one imaginary root of the equation

$$x^n - 1 = 0,$$

then will every power of a likewise be a root.

For, since $a^n = 1$, $a^{2n} = 1$, $a^{3n} = 1$, &c.; therefore,

$$a^2,\ a^3,\ a^4,\ \&c.,$$

are roots of the equation.

The roots of the equation may therefore be represented by the various terms of the series,

$$\ldots a^{-3},\ a^{-2},\ a^{-1},\ 1,\ a,\ a^2 \ldots\ldots \&c.,$$

in which, however the terms may differ in form, they cannot present more than n values, otherwise the equation would have more than n roots.

178. If a be one imaginary root of $x^n + 1 = 0$, then will every odd power of a likewise be a root.

For, since $a^n = -1$, every odd power of a^n will be equal to -1; consequently, the different terms of the series

$$a^{-5},\ a^{-3},\ a^{-1},\ a\ a^{3},\ a^{5},\ \&c.,$$

will all be roots of the equation

$$x^n + 1 = 0.$$

179. If n is a prime number, and a one root of the equation

$$x^n - 1 = 0,$$

then will

$$1,\ a,\ a^2 \ldots\ldots a^{n-1}$$

be all different, and therefore will form the complete series of roots.

For, if possible, let $a^p = a^q$, p and q both being less than n, then

$$a^{p-q} = 1,$$

in which $p - q$ is less than n.

Now, let $p - q$ be contained in n r times, leaving a remainder p', which will be less than $p - q$,

then $\qquad a^n = a^{r(p-q)+p'} = a^{r(p-q)} \,.\, a^{p'} = 1$

$\therefore \qquad a^{p'} = 1,$

If p'' be the remainder arising from dividing n by p', we will have in like manner

$$a^{p''} = 1.$$

Proceeding in this manner, we shall finally arrive at the equation

$$a = 1,$$

which is manifestly absurd.

Therefore, $a^p = a^q$ is impossible.

Since, then, $1.\ a,\ a^2 \ldots\ldots a^{n-1}$ are all different, and each is a root, they must form the complete series of roots.

Cor. Since $a^n = 1$, we will likewise have

$$a^n,\ a^{n+1} \ldots\ldots a^{2n-1}$$

for the roots.

To illustrate the above theory, let it be required to determine the roots of

$$x^5 - 1 = 0.$$

One root is 1, therefore, dividing by $x - 1 = 0$ the equation becomes

$$x^4 + x^3 + x^2 + x + 1 = 0,$$

a recurring equation of the fourth order, whose roots may be found by (Art. 166.)

180. If p and q are prime to each other, then $x^p - 1 = 0$ $x^q - 1 = 0$, have no common root except 1.

For, if it be possible, let a be a common root, so that

$$a^p = 1, a^q = 1.$$

Let $p = mq + q'$, in which q' is less than q,

then $$a^p = a^{mq} \times a^{q'} = a^{q'} = 1.$$

In like manner, if $q = nq' + q''$, we will have

$$a^{q''} = 1,$$

q'' being less than q'. Proceeding thus, we will finally arrive at the equation

$$a = 1,$$

which is impossible.

181. If $n = p. q. r$, (p, q and r being prime numbers,) then the roots of $x^n = 1$ will be the roots of $x^p = 1$, $x^q = 1$ and $x^r = 1$.

For $$x^n = x^{pqr} = (x^p)^{qr} = 1,$$

$\therefore$ $$x^p = 1,$$

similarly $$x^q = 1,$$

and $$x^r = 1.$$

The roots of these equations will therefore satisfy the equation

$$x^n - 1 = 0.$$

182. When n is the product of two prime numbers, p and q, the roots of $x^n - 1 = 0$, will be expressed by the products arising by multiplying every root of $x^p - 1 = 0$ by every root of $x^q - 1 = 0$.

Let the roots of $x^p - 1 = 0$ be

$$1, a, a^2, a^3 \ldots . a^{p-1},$$

and those of $x^q - 1 = 0$ be

$$1, b, b^2, b^3 \ldots . a^{q-1},$$

then since $(a^h)^n = 1$, and $(b^k)^n = 1$,

therefore $(a^h b^k)^n = 1$, and $a^h b^k$ is a root

of the equation $x^n = 1$, or $x^n - 1 = 0$.

Moreover, these products are all different, for, if possible, let $a^h b^k = a^g b^l$, then we will have $a^{h-g} = b^{l-k}$,
but a^{h-g} is a root of $x^p - 1 = 0$,
and b^{l-k} is a root of $x^q - 1 = 0$.
The two equations have therefore a common root, but this is impossible, (Art. 180.)

Since, then, all the products are roots of $x^n - 1 = 0$, and since no two are equal, and their number is $pq = n$, they form the complete series of roots of the given equation.

If $p = q$, the above demonstration fails to give the series of roots.

In this case let the roots of $x^p - 1 = 0$ be

$$1, a, a^2, a^3 \dots . a^{p-1}.$$

If, then, we form the series of equations

$$x^p = 1, \; x^p = a, \; x^p = a^2, \; x^p = a^3, \text{ \&c.},$$

the pp roots of these equations will be roots of $x^n - 1 = 0$, and as they will evidently be all different, they comprise the complete series of roots.

Now the roots of $x^p = a^g$ are (Art. 174) equal to those of $x^p = 1$, multiplied by $\sqrt[p]{a^g}$.

The complete series of roots of $x^n - 1 = 0$, is therefore

$$1, a, a^2 \dots . a^{p-1}$$
$$\sqrt[p]{a}, a\sqrt[p]{a}, a^2\sqrt[p]{a} \dots . a^{p-1}\sqrt[p]{a}$$
$$\sqrt[p]{a^2}, a\sqrt[p]{a^2}, a^2.\sqrt[p]{a^2} \dots . a^{p-1}\sqrt[p]{a^2}.$$
&c.

CHAPTER IX.

SERIES.

SECTION I.

Method of Indeterminate Coefficients.

183. If there are two series,

$$Ax^a + Bx^b + Cx^c + \&c.,$$

and

$$Mx^m + Nx^n + Px^p + \&c.,$$

in which the indices are arranged in the order of their magnitudes, beginning at the least, and which are equal, whatever value may be assigned to x; the indices and likewise the coefficients of the corresponding terms must be equal.

For if a be not equal to m, one of them, as a, must be the greater. Dividing by x^m we shall have

$$Ax^{a-m} + Bx^{b-m} + Cx^{c-m} + \&c., = M + Nx^{n-m} + Px^{p-m} + \&c.,$$

whatever value we assign to x. But if $x = 0$ all the terms except the first in the second series will vanish, we shall therefore have $M = 0$; but this is impossible. Hence

$$a = m,$$

and the equation above becomes

$$A + Bx^{b-a} + Cx^{c-a} + \&c., = M + Nx^{n-a} + Px^{p-a} + \&c.$$

If in this $x = 0$, we shall have

$$A = M.$$

Again, since the first terms of the series are identical, we have also

$$Bx^b + Cx^c + \&c., = Nx^n + Px^p + \&c.$$

The same reasoning will evidently show that

$$b = n, \text{ and } B = N,$$

and so for the other coefficients and indices.

184. If a series

$$Ax^a + Bx^b + Cx^c + \&c.,$$

in which the indices are arranged as before, be equal to zero, whatever value may be assigned to x, each of the terms must be equal to zero.

For since $$Ax^a + Bx^b + Cx^c + \&c. = 0,$$
we have $$A + Bx^{b-a} + Cx^{c-a} + \&c. = 0.$$

In which, if we substitute 0 for x, we will have $A = 0$, and, consequently,

$$Bx^b + Cx^c + \&c. = 0.$$

If we divide this by x^b, and make $x = 0$, we shall have

$$B = 0,$$

and so for the rest.

These two propositions being of great importance in the development of functions and the summation of series, we shall illustrate them pretty fully by examples. In applying the method we are generally able to detect the nature of the indices, and thus avoid the trouble of calculating them.

Examples.

Ex. 1. Let it be required to develope $\frac{1}{1+x}$ in a series.

Here if $x = 0$, the value of the function is 1. Consequently, the first term of the series is 1. The other terms can have no negative indices. For if one of them is of the form Mx^{-n}, or $\frac{M}{x^n}$, this would be infinite, when $x = 0$.

In order to make the demonstration general, we will assume

$$\frac{1}{1+x} = 1 + Ax^a + Bx^b + Cx^c + Dx^d + \&c.$$

Clearing of fractions and transposing, we have

$$1 - x - Ax^{a+1} - Bx^{b+1} - Cx^{c+1}$$
$$= 1 + Ax^a + Bx^b + Cx^c + Dx^d.$$

Equating the exponents, we have

$$a = 1,\ b = a + 1 = 2,\ C = b + 1 = 3,\ d = 4,\ \&c.$$

Also, $A = -1$, $B = -A = 1$, $C = -B = -1$ $D = 1$.

Substituting these values, we obtain

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \&c.,$$

which may be verified by division.

Ex. 2. Develope $\frac{1+x}{1-x}$.

Assume $\frac{1+x}{1-x} = 1 + Ax^a + Bx^b + Cx^c + \&c.$

Clearing of fractions and transposing, we have

$$1+2x+Ax^{a+1}+Bx^{b+1}+Cx^{c+1}+\&c.$$
$$= 1+Ax^a+Bx^b+Cx^c+Dx^d+\&c.$$

Whence $a = 1,\ b = 2,\ c = 3,\ \&c.,$

and $A = 2,\ B = A = 2,\ C = 2,\ \&c.,$

$$\therefore \quad \frac{1+x}{1-x} = 1+2x+2x^2+2x^3+\&c.$$

Ex. 3. Develope $\frac{1}{(1+x)^2}$.

Assume $\frac{1}{(1+x)^2}$ or $\frac{1}{1+2x+x^2} = 1 + Ax + Bx^2 + Cx^3$ $+ Dx^4 + \&c.$

Clearing of fractions we have

$$\begin{aligned} 1 = 1 + Ax + {} & Bx^2 + Cx^3 + Dx^4 + \&c. \\ +2x & + 2Ax^2 + 2Bx^3 + 2Cx^4 + \&c. \\ & + x^2 + Ax^3 + Bx^4 + \&c. \end{aligned}$$

or

$$\begin{aligned} 0 = Ax + {} & Bx^2 + Cx^3 + Dx^4 + \&c. \\ 2x & + 2Ax^2 + 2Bx^3 + 2Cx^4 + \&c. \\ & + x^2 + Ax^3 + Bx^4 + \&c. \end{aligned}$$

Whence

$$\begin{aligned} A + 2 &= 0 \text{ or } A = -2 \\ B + 2A + 1 &= 0 \text{ or } B = 3 \\ C + 2B + A &= 0 \text{ or } C = -4 \\ D + 2C + B &= 0 \text{ or } D = 5, \&c. \end{aligned}$$

$$\therefore \quad \frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \&c.$$

Ex. 4. Develope $\frac{a^3}{a^2 - x^2}$.

Assume this equal to the series

$$a + Ax^n + Bx^b + Cx^c + Dx^d + \&c.,$$

and clear of fractions. Then

$$\begin{aligned} a^3 = a^3 + Aa^2x^n &+ Ba^2x^b + Ca^2x^c + \&c. \\ - ax^2 &- Ax^{n+2} - Bx^{b+2} + \&c. \end{aligned}$$

Whence $n = 2,\ b = n + 2 = 4,\ c = b + 2 = 6,$

also $\quad Aa^2 = a$ or $A = \frac{1}{a}$

$$Ba^2 = A \text{ or } B = \frac{1}{a^3}$$

$$Ca^2 = B \text{ or } C = \frac{1}{a^5}.$$

Hence $\quad \frac{a^3}{a^2 - x^2} = a + \frac{x^2}{a} + \frac{x^4}{a^3} + \frac{x^6}{a^5} + \frac{x^8}{a^7}$, &c.

Ex. 5. Develope $\sqrt{a^2 + x^2}$.

Assume $\sqrt{a^2 + x^2} = a + Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$

$$\begin{aligned}\text{Squaring } a^2+x^2 = a^2 &+ Aax + Bax^2 + Cax^3 + Dax^4 + \&c.\\ &+ Aax + A^2x^2 + ABx^3 + ACx^4 + \&c.\\ &\qquad + Bax^2 + ABx^3 + B^2x^4 + \&c.\\ &\qquad\qquad + Cax^3 + ACx^4 + \&c.\\ &\qquad\qquad\qquad + Dax^4 + \&c.\end{aligned}$$

$$\begin{aligned}\text{Whence} \quad 2Aa &= 0 & \text{or } A &= 0\\ 2Ba &= 1 & \text{or } B &= \frac{1}{2a}\\ 2Ca &= 0 & C &= 0\\ 2Da &= -B^2 & \text{or } D &= -\frac{1}{8a^3},\end{aligned}$$

and $\sqrt{a^2 + x^2} = A + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \&c.$

The operation would have been shortened had we assumed

$$\sqrt{a^2 + x^2} = a + Ax^2 + Bx^4 + Cx^6 + Dx^8 + \&c.$$

Squaring we have

$$\begin{aligned}a^2 + x^2 = a^2 &+ Aax^2 + Bax^4 + Cax^6 + Dax^8 + \&c.\\ &+ Aax^2 + A^2x^4 + ABx^6 + ACx^8 + \&c.\\ &\qquad + Bax^4 + ABx^6 + B^2x^8 + \&c.\\ &\qquad\qquad + Cax^6 + CAx^8 + \&c.\\ &\qquad\qquad\qquad + Dax^8 + \&c.\end{aligned}$$

Whence $2\,Aa = 1 \qquad \text{or } A = \dfrac{1}{2\,a}$

$2\,Ba = -A^2 \qquad \text{or } B = -\dfrac{1}{8\,a^3}$

$2\,Ca = -2\,AB \qquad \text{or } C = \dfrac{1}{16\,a^5}$

$2\,Da = -2\,AC - B^2 \text{ or } D = -\dfrac{5}{128\,a^7}.$

Hence $\sqrt{a^2 + x^2} = a^2 + \dfrac{x^2}{2\,a} - \dfrac{x^4}{8\,a^3} + \dfrac{x^6}{16\,a^5} - \dfrac{5.x^8}{128\,a^7} +$ &c.

This series is defective since the law of variation of the terms is not manifest. If we write it as follows, this defect will be remedied, though the law can still hardly be considered as demonstrated.

$$\sqrt{a^2 - x^2} = a^2 + \frac{x^2}{2\,a} - \frac{x^4}{2.4\,a^3} + \frac{3\,a^6}{2.4.6\,a^5} - \frac{3.5\,x^8}{2.4.6.8\,a^7} + \&c.$$

Ex. 6. Develope $\dfrac{1 - x}{1 + x + x^2}$.

Ans. $1 - 2\,x + x^2 + x^3 - 2\,x^4 + x^5 + x^6 - 2\,x^7 + x^8 + \&c.$

Ex. 7. Develope $\dfrac{1 + 3\,x}{1 - 2\,x + x^2}$.

Ans. $1 + 5\,x + 9\,x^2 + 13\,x^3 + 17\,x^4 + \&c.$

Ex. 8. Develope $\dfrac{x^2 + a^2}{x^2 - a^2}$.

Ans. $1 + 2\dfrac{a^2}{x^2} + 2\dfrac{a^4}{x^4} + 2\dfrac{a^6}{x^6} + \&c.$

Ex. 9. Develope $\dfrac{1 + 2\,x}{1 - x - x^2}$.

Ans. $1 + 3\,x + 4\,x^2 + 7\,x^3 + 11\,x^4 + \&c.$

Ex. 10. Develope $\dfrac{a + 2\,x}{(1 - x)^3}$.

Ans.

185. Develope $(1+x)^n$.

Assume $(1+x)^n = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$

If we square both sides, we will have

$$\begin{array}{lllllll}
(1+x)^{2n} = 1 & + Ax + & Bx^2 + & Cx^3 + & Dx^4 + & \&c., \\
 & + Ax + & A^2x^2 + & ABx^3 + & ACx^4 & \text{"} \\
 & & + Bx^2 + & ABx^3 + & B^2x^4 & \text{"} \\
 & & & + Cx^3 + & ACx^4 & \text{"} \\
 & & & & + Dx^4 & \text{"}
\end{array}$$

This series is evidently perfect so far as the last column inclusive.

Now, $(1+x)^{2n} = \{(1+x)^2\}^n = \{1+(2x+x^2)\}^n$; and since the coefficients A, B, &c., are entirely independent of x, we shall have

$$\{1+(2x+x^2)\}^n = 1 + A(2x+x^2) + B(2x+x^2)^2 + C(2x+x^2)^3 + \&c.$$

$$\begin{array}{llll}
= 1 + 2Ax + & Ax^2 & & \\
 & + 4Bx^2 + & 4Bx^3 + & Bx^4 \\
 & & + 8Cx^3 + & 12Cx^4 +, \&c. \\
 & & & + 16Dx^4 +, \&c.
\end{array}$$

Now, as this series and the former must be identical, we have

$$2A = 2A,$$

$$2B + A^2 = 4B + A \text{ or } B = \frac{A(A-1)}{2}$$

$$2C + 2AB = 8C + 4B \text{ or } C = \frac{B(A-2)}{3} = \frac{A.(A-1).(A-2)}{2.3}$$

$$2D + 2AC + B^2 = 16D + 12C + B \text{ or } D = \frac{A.(A-1)(A-2)(A-3)}{2.3.4},$$

whence the law of the coefficients is plain.

186. It only remains to determine A, which may be done as follows.

1st. Let n be a positive integer.

If we multiply the equation

$$(1+x)^n = 1 + Ax + \&c.,$$

by $(1+x)$ we shall have

$$(1+x)^{n+1} = 1 + (A+1)x + \&c.$$

From this we perceive that increasing the index by unity increases the coefficient of the second term of the development by unity. Hence, as the index and coefficient of the second term are the same in the first power, since $(1+x)^1=1+x$ they will also be for any positive integral value of n.

In this case, therefore,

$$(1+x)^n = 1 + nx + \&c.$$

2d. Let n be a positive fraction as $\frac{p}{q}$; p and q being positive integers, then we have

$$(1+x)^{\frac{p}{q}} = 1 + Ax + \&c.$$

Raise both members to the qth power. To facilitate this operation, we may observe that as every term subsequent to the second will contain the square or higher powers of x; these cannot affect the coefficient of the second term in the power. We may, therefore, so far as this term is concerned, consider

$$(1+x)^{\frac{p}{q}} = 1 + Ax.$$

Hence $\quad (1+x)^p = (1+Ax)^q = 1 + qAx + \&c.$

But $\quad (1+x)^p = \quad 1 + px + \&c.,$

$\therefore \quad qA = p$ or $A = \frac{p}{q}$.

3d. Let n be a negative integer or fraction, and equal to $-m$.

Then $(1+x)^n = \frac{1}{(1+x)^m} = \frac{1}{1+mx+\&c.}$ = (by actual division,) $1 - mx + \&c.,$

$$(1+x)^{-m} = 1 + Ax + \&c.,$$

and $\quad A = -m.$

In all cases, therefore, we will have the coefficient of the second term equal to the index of the power.

The coefficients are therefore

$$A = n$$

$$B = n.\frac{n-1}{2}$$

$$C = n.\frac{n-1}{2}.\frac{n-2}{3}, \&c.$$

and $(1+x)^n = 1 + nx + n.\frac{n-1}{2}x^2 + n.\frac{n-1}{2}.\frac{n-2}{3}x^3 + \&c.$

187. Develope $(a+x)^n$.

It is evident that $a+x=a\left(1+\frac{x}{a}\right)$,

and therefore $\quad (a+x)^n=a^n\left(1+\frac{x}{a}\right)^n$.

Substituting $\frac{x}{a}$ for x in the development in last question, it becomes

$$\left(1+\frac{x}{a}\right)^n=1+n\frac{x}{a}+n.\frac{n-1}{2}.\frac{x^2}{a^2}+n.\frac{n-1}{2}.\frac{n-2}{3}\frac{x^3}{a^3}+\&c.$$

$$\therefore\ (a+x)^n=a^n+na^{n-1}x+n.\frac{n-1}{2}.\ a^{n-2}x^2+n.\ \frac{n-1}{2}.\frac{n-2}{3}.$$
$$a^{n-3}x^3+\ \&c.$$

This is Newton's celebrated Binomial Theorem.

If n is a positive integer, we shall finally arrive at a term which will contain the factor $n-n$, and which, of course, vanishes, as will all the following terms of the general series. In all other cases the series will continue to infinity.

188. This demonstration fails in one important point, for though we may extend the calculation of the coefficients as far as we please, and still find them correct, yet as the equations from which they are determined become more and more complex as we proceed, we are still in doubt whether some yet uncalculated may not vary from the rule which has apparently been established. We should be very careful to avoid generalizing a result unless we can prove it to be general. A want of attention to this principle has frequently led to important errors.

The following demonstration has not this fault.

Assume $\quad (1+x)^n=1+Ax+Bx^2+Cx^3+\ \&c.$,

then $\quad (1+y)^n=1+\ Ay+By^2+Cy^3+\ \&c.$,

$\therefore (1+x)^n-(1+y)^n=A(x-y)+B(x^2-y^2)+C(x^3-y^3)+\&c.$

whence

$$\frac{(1+x)^n-(1+y)^n}{x-y}=A+B\frac{x^2-y^2}{x-y}+C\,\frac{x^3-y^3}{x-y}+D\frac{x^4-y^4}{x-y}+\&c.$$

Put $1+x=v$ and $1+y=w$, then $x-y=v-w$, and we shall have

$$\frac{v^n - w^n}{v - w} = A + B\frac{x^2 - y^2}{x - y} + C\frac{x^3 - y^3}{x - y} + D\frac{x^4 - y^4}{x - y} + \&c.$$

If now we make $v = w$ or $x = y$, this will become (see next article)

$$nv^{n-1} = A + 2\,Bx + 3\,Cx^2 + 4\,Dx^3 + \&c.,$$

hence $\quad nv^n = A + 2\,Bx + 3\,Cx^2 + 4\,Dx^3 + \&c.,$

$\qquad\qquad + \quad Ax + 2\,Bx^2 + 3\,Cx^3 + \&c.,$

but $\quad nv^n = n + nAx + nBx^2 + nCx^3 + \&c.$

$\therefore \quad A = n,$

$$2B + A = nA \text{ or } B = A.\frac{n-1}{2},$$

$$3C + 2B = nB \text{ “ } C = B.\frac{n-2}{3},$$

$$4D + 3C = nC \text{ or } D = C.\frac{n-3}{4}.$$

in which the law of the equations is at once manifest from the series.

Substituting in each of these values that of the preceding coefficient, they become

$$A = n$$

$$B = n.\frac{n-1}{2},$$

$$C = n.\frac{n-1}{2}.\frac{n-2}{3},$$

$$D = n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4},$$

&c.

as before.

189. If n is a whole positive number

$$\frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 \ldots\ldots x^2y^{n-3} + xy^{n-2}$$
$$+ y^{n-1},$$

the number of terms being n. This may be proved by simple division. Now, as this is true for every value of x and y; let $x = y$ and its value becomes

$$nx^{n-1}.$$

Next let n be a positive fraction $= \frac{p}{q}$.

Put $x = v^q$, so that $x^{\frac{p}{q}} = v^p$,

and $y = w^q$, or $y^{\frac{p}{q}} = w^p$.

By the substitution of these values the fraction

$\dfrac{x^{\frac{p}{q}} - y^{\frac{p}{q}}}{x - y}$ becomes changed into

$$\frac{v^p - w^p}{v^q - w^q} = \frac{\dfrac{v^p - w^p}{v - w}}{\dfrac{v^q - w^q}{v - w}}.$$

But since p and q are positive integers, we shall have, when $x = y$ or $v = w$,

$$\frac{\dfrac{v^p - w^p}{v - w}}{\dfrac{v^q - w^q}{v - w}} = \frac{pv^{p-1}}{qv^{q-1}} = \frac{p}{q} v^{p-q}.$$

Now since $v = x^{\frac{1}{q}}$, $v^{p-q} = x^{\frac{p-q}{q}} = x^{\frac{p}{q}-1}$,

$\therefore$ when $x = y$, we shall have

$$\frac{x^{\frac{p}{q}} - y^{\frac{p}{q}}}{x - y} = \frac{p}{q} x^{\frac{p}{q}-1}.$$

Lastly, let n be a negative number, whole or fractional, we shall have $\dfrac{x^{-n} - y^{-n}}{x - y} = \dfrac{x^{-n}y^{-n}(y^n - x^n)}{x - y}$

$$= -x^{-n}y^{-n}\frac{x^n - y^n}{x - y}.$$

When $x = y$, this becomes equal to

$$-x^{-n}y^{-n} . nx^{n-1}$$
$$= -nx^{-n-1}.$$

This point being thus established, the preceding demonstration becomes complete.

190. If we examine the formula

$(a + x)^n = a^n + na^{n-1}x + n . \dfrac{n-1}{2} . a^{n-2}x^2 + n . \dfrac{n-1}{2} . \dfrac{n-2}{3} a^{n-3}x^3, + \&c.,$

we shall at once perceive that any coefficient may be derived from that of the preceding term, by multiplying by the exponent of the leading factor in that term, and dividing by the number of terms to that place.

Thus the coefficient of the fourth term is found by multiplying

$$n \cdot \frac{n-1}{2},$$

by $n-2$, and dividing by 3.

EXAMPLES.

Ex. 1. Raise $(a-x)$ to the seventh power.

The operation is as follows:

$$(a-x)^7 =$$
$$a^7 - 7a^6x + 21a^5x^2 - 35a^4x^3 + 35a^3x^4 - 21a^2x^5 + 7ax^6 - x^7$$

$$\begin{array}{ccc} 6 & 5 & 4 \\ 2\,)\,\overline{42} & 3\,)\,\overline{105} & 4\,)\,\overline{140} \\ \overline{21} & \overline{35} & \overline{35} \end{array}$$

x being negative, its odd powers will also be negative, and the signs alternate as above.

Ex. 2. Develope $\sqrt{a-x}$.

Here $n = \frac{1}{2}$, and the several coefficients will be

$$n = \frac{1}{2} \qquad = \frac{1}{2}$$

$$n \cdot \frac{n-1}{2} = -\frac{1}{8} \qquad = -\frac{1}{2.4}$$

$$n \cdot \frac{n-1}{2} \cdot \frac{n-2}{4} = +\frac{1}{16} = \frac{3}{2.4.6}$$

$$n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} = -\frac{3.5}{2.4.6.8}.$$

Hence $\sqrt{a-x} = a^{\frac{1}{2}} - \frac{1}{2}a^{-\frac{1}{2}}x - \frac{1}{2.4}a^{-\frac{3}{2}}x^2 - \frac{3}{2.4.6}a^{-\frac{5}{2}}x^3$

$- \frac{3.5}{2.4.6.8}a^{-\frac{7}{2}}x^4 -$ &c.

$$= a^{\frac{1}{2}} - \frac{x}{2.a^{\frac{1}{2}}} - \frac{x^2}{2.4\,a^{\frac{3}{2}}} - \frac{3\,x^3}{2.4.6\,a^{\frac{5}{2}}} - \frac{3.5\,x^4}{2.4.6.8\,x^{\frac{7}{2}}} - \text{\&c.}$$

Ex. 3. Develope $\dfrac{a}{(a^2+x^2)^2}$.

This function is equivalent to $a\,(a^2+x^2)^{-2}$.

The terms of the development are as follows:

$$\begin{array}{lcl}
(a^2)^n = & a^{-4} = & \dfrac{1}{a^4} \\
n\,(a^2)^{n-1}(x^2) = & -2\,a^{-6}x^2 = & -\dfrac{2\,x^2}{a^6} \\
n\,.\dfrac{n-1}{2}(a^2)^{n-2}(x^2)^2 = & +3\,a^{-8}x^4 = & \dfrac{3\,x^4}{a^8} \\
n.\dfrac{n-1}{2}.\dfrac{n-2}{3}(a^2)^{n-3}(x^2)^3 = & -4\,a^{-10}x^6 = & -\dfrac{4\,x^6}{a^{10}} \\
\&c. & = & \&c.
\end{array}$$

$$\therefore\quad (a^2+x^2)^{-2} = \frac{1}{a^4} - \frac{2\,x^2}{a^6} + \frac{3\,x^4}{a^8} - \frac{4\,x^6}{a^{10}} + \&c.,$$

$$\text{and}\quad \frac{a}{(a^2+x^2)^2} = \frac{1}{a^3} - \frac{2\,x^2}{a^5} + \frac{3\,x^4}{a^7} - \frac{4\,x^6}{a^9} + \&c.$$

Ex. 4. What is the 4th power of $(a - x)$?

Ans. $a^4 - 4\,a^3x + 6\,a^2x^2 - 4\,ax^3 + x^4$.

Ex. 5. What is the 7th power of $(b + c)$?

Ans. $b^7 + 7\,b^6c + 21\,b^5c^2 + 35\,b^4c^3 + \&c.$

Ex. 6. What is the 3d power of $(3 + x)$?

Here we shall have

$$\begin{aligned}(3+x)^3 &= 3^3 + 3.3^2x + 3.3\,x^2 + x^3 \\ &= 27 + \ 27\,x + \ 9\,x^2 + x^3.\end{aligned}$$

Ex. 7. What is the 5th power of $(x + 2)$?

Ans. $x^5 + 10\,x^4 + 40\,x^3 + 80\,x^2 + 80\,x + 32$.

Ex. 8. What is the 7th power of $(2\,b + x)$?

Ans. $128\,b^7 + 448\,b^6x + 672\,b^5x^2 + 560\,b^4x^3 + 280\,b^3x^4$
$+ 84\,b^2x^5 + 14\,bx^6 + x^7$.

Ex. 9. Develope $\sqrt{a^2+x}$.

Ans. $a+\frac{1}{2}\frac{x}{a}-\frac{x^2}{2.4.a^3}+\frac{3\,x^3}{2.4.6\,a^5}+\frac{3.5\,x^4}{2.4.6.8\,a^7}+\text{\&c.}$

$$=a+\frac{x}{2\,a}-\frac{x^2}{8\,a^3}+\frac{x^3}{16\,a^5}-\frac{5\,x^4}{128\,a^7}+\text{\&c.}$$

Ex. 10. Develope $\sqrt[3]{b^3+x}$.

Result. $b+\frac{x}{3.b^2}-\frac{2\,x^2}{3.6\,b^5}+\frac{2.5\,x^3}{3.6.9\,b^8}-\frac{2.5.8\,x^4}{3.6.9.12\,b^{11}}+\text{\&c.}$

Ex. 11. Develope $\sqrt[4]{(c^2-x^2)^3}$ or $(c^2-x^2)^{\frac{3}{4}}$.

Result. $c^{\frac{3}{2}}\left\{1-\frac{3\,x^2}{2^2c^2}-\frac{3\,x^4}{2^5c^4}-\frac{5\,x^6}{2^7c^6}-\frac{45x^8}{2^{11}.c^8}-\text{\&c.}\right\}$

Ex. 12. Develope $\sqrt{\frac{a-x}{a+x}}$.

This expression is equivalent to

$$\sqrt{\frac{(a-x)^2}{a^2-x^2}}=(a-x)\,.\,(a^2-x^2)^{-\frac{1}{2}}.$$

$$(a^2-x^2)^{-\frac{1}{2}}=a^{-1}+\frac{1}{2}a^{-3}x^2-\frac{3}{2.4}a^{-5}x^4-\frac{3.5}{2.4.6}a^{-7}x^6$$

$$-\frac{3.5.7}{2.4.6.8}a^{-9}x^8-\text{\&c.}$$

$$=\frac{1}{a}+\frac{x^2}{2.a^3}-\frac{3\,x^4}{2.4.a^5}-\frac{3.5.x^6}{2.4.6\,a^7}-\frac{3.5.7\,x^8}{2.4.6.8\,a^9}.$$

$$\therefore\ \sqrt{\frac{a-x}{a+x}}=(a-x)\left\{\frac{1}{a}+\frac{x^2}{2.a^3}-\frac{3\,x^4}{2.4\,a^5}-\frac{3.5.x^6}{2.4.6\,a^7}-\text{\&c.}\right\}$$

SECTION II.

The Differential Method,

Or the method of determining the successive differences of the terms of a series, and thence any intermediate term, and the sum of the terms of the series.

1. Let $a, b, c, d, e, f, g, \ldots$ be any series. The first order of differences is evidently

$$b-a, \quad c-b, \quad d-c, \text{ and } e-d, \text{ \&c.}$$

If in like manner we take the differences of the successive terms of this series, we shall have the second order of differences, as follows:

$$c-2b+a, \quad d-2c+b, \quad e-2d+c, \text{ \&c.}$$

The third order will in like manner be

$$d-3c+3b-a, \quad e-3d+3c-b.$$

The fourth

$$e-4d+6c-4b+a.$$

If, now, we examine the coefficients of the several terms in the differences of the various orders, we shall at once perceive that they correspond with those of the different powers of a binomial. A very slight attention to the mode in which the successive differences are formed, will convince us that this coincidence must hold good, whatever be the order of differences.

We may therefore conclude that the first term of the nth order of differences will be, if n be even,

$$a-n.b+n.\frac{n-1}{2}c-n.\frac{n-1}{2}.\frac{n-2}{3}d+\text{\&c.}$$

And if n be odd,

$$-a+n.b-n.\frac{n-1}{2}c+n.\frac{n-1}{2}.\frac{n-2}{3}d-\text{\&c.}$$

EXAMPLES.

Ex. 1. Required the first term of the third order of differences in the series of cubes,

1, 8, 27, 64, 125.

As n is odd we use the second series, and the term is

$$-1+3.8-3.27+64=6.$$

If the first term of the fourth order were required, it would be equal to

$$1-4.8+6.27-4.64+125$$
$$=1-32+162-256+125=0.$$

Ex. 2. Required the first term of the fifth order of differences in the series

$$1, 2, 2^2, 2^3, 2^4, \&c.$$

The second formula gives

$$-1+5.2-10.2^2+10.2^3-5.2^4+2^5=1.$$

Ex. 3. What is the first term of the fourth order of differences in the series

$$1, 2^4, 3^4, 4^4, 5^4 \ldots\ldots ?$$

Ans. 24.

Ex. 4. What is the first term of the fifth order of differences in the series

$$1, 5, 15, 35, 70, 126, \&c.?$$

Ans. 0.

Ex. 5. Required the first term of the fifth order of differences of the series

$$1, 6, 21, 56, 126, 252, 462, \&c.$$

Ans. 1.

192. Let it now be required to find the nth term of the series $a, b, c, d, \ldots$

If we represent by d_1, d_2, d_3, d_4, &c., the first term of the 1st, 2d, 3d, 4th, &c., order of differences, we shall have

$$\left.\begin{array}{l} d_1=-a+b \\ d_2=a-2b+c \\ d_3=-a+3b-3c+d \\ d_4=a-4b+6c-4d+e \end{array}\right\} \text{whence} \left\{\begin{array}{l} b=a+d_1 \\ c=-a+2b+d_2 \\ d=a-3b+3c+d_3 \\ e=-a+4b-6c+4d+d_4 \end{array}\right.$$

or

$$b=a+d_1,$$
$$c=a+2d_1+d_2,$$
$$d=a+3d_1+3d_2+d_3,$$
$$e=a+4d_1+6d_2+4d_3+d_4,$$

and the nth term $=a+(n-1)d_1+(n-1)\dfrac{n-2}{2}d_2+$ &c.

This series will terminate if the differences vanish after a certain order. If they do not, it will be infinite, and the nth term can only be found approximately.

Examples.

Ex. 1. What is the 12th term of the order of cubes? in other terms, what is 12^3?

Here $n = 12$, $a = 1$, $d_1 = 7$, $d_2 = 12$, $d_3 = 6$, $d_4 = 0$,

$$\therefore \quad 12^3 = 1 + 11.7 + 11.\frac{10}{2}.12 + \frac{11.10.9}{2.3}.6$$

$$= 1 + 77 + 660 + 990 = 1728.$$

Ex. 2. What is the 50th term of the series

1 . 4 . 8 . 13 . 19 . . &c.?

Here $a = 1$, $d_1 = 3$, $d_2 = 1$, $d_3 = 0$,

and the 50th term $= 1 + 49.3 + \frac{49.48}{2}$,

$$= 1 + 147 + 1176 = 1324.$$

Ex. 3. Required the 20th term of the series 1, 5, 15, 35, 70.126, &c. *Ans.* 8855.

Ex. 4. What is the 30th term of the series, 1, 3, 6, 10, 15, &c.? *Ans.* 465.

Ex. 5. What is the 20th term of the series

1, 6, 21, 56, 126, 252, 462, &c.?

Ans. 42504.

193. Required the sum of n terms of the series

$$a, b, c, d, \ldots\ldots$$

Now, it is evident this is the same as the $(n+1)$th term of the series

$$0, 0 + a, a + b, a + b + c, a + b + c + d, \text{\&c.},$$

in which the first terms of the various orders of difference are a, d_1, d_2, d_3, &c.

Hence $a+b+c+d\ldots = na + n.\frac{n-1}{2}d_1 + n.\frac{n-1}{2}.\frac{n-2}{3}d_2 +$ &c.

EXAMPLES.

Ex. 1. What is the sum of n terms in the series

1, 4, 7, 10, &c.?

Here $a=1,\ d_1=3,\ d_2=0,$

and $S=n+3.\ n.\dfrac{n-1}{2}=\dfrac{3n^2-n}{2}.$

Ex. 2. What is the sum of n terms in the series

1, 3, 5, 7, &c.?

Here $a=1,\ d_1=2,$

and $S=n+n.\dfrac{n-1}{2}.\ 2=n^2.$

Ex. 3. What is the sum of n terms of the series

$1,\ 2^2,\ 3^2,\ 4^2$, &c.?

Here $a=1,\ d_1=3,\ d_2=2,\ d_3=0,$

$$\therefore S=n+n.\frac{n-1}{2}.3+n.\frac{n-1}{2}.\frac{n-2}{3}.2=n+\frac{3n^2-3n}{2}+\frac{n^3-3n^2+2n}{3}$$

$$=\frac{2n^3+3n^2+n}{6}=\frac{n.(n+1)(2n+1)}{6}.$$

Ex. 4. What is the sum of n terms of the series of cubes?

Ans. $\dfrac{n^2(n+1)^2}{4}.$

Ex. 5. What is the sum of the series $1,\ 2^4,\ 3^4$, &c.?

Ans. $\dfrac{n^5}{5}+\dfrac{n^4}{2}+\dfrac{n^3}{3}-\dfrac{n}{30}.$

Ex. 6. What is the sum of 12 terms of the series

1, 4, 8, 13, 19?

Ans. 430.

Ex. 7. What is the sum of 10 terms of the series

1, 5, 15, 35, 70, 126, &c.?

Ans. 2002.

SECTION III.

On the Summation of Infinite Series.

194. An infinite series is one the number of whose terms is unlimited; the law of succession being generally discoverable by the examination of a few terms.

195. A *converging* series is one the successive terms of which become smaller and smaller, as

$$1 \cdot \tfrac{1}{2} \cdot \tfrac{1}{4} \cdot \tfrac{1}{8} \cdot \tfrac{1}{16} \cdot \cdot \cdot \cdot$$

196. A *diverging* series is one whose successive terms become greater and greater, as

$$1, 4, 16, 64 \ldots .$$

197. An *ascending* series is one in which the powers of the unknown become greater as we proceed.

Thus $$a, bx\ cx^2\ dx^3, \&c.,$$
is an ascending series.

198. A descending series has its powers diminishing as the series proceeds: as

$$a, \frac{b}{x}, \frac{c}{x^2}, \frac{d}{x^3}, \&c.$$

199. As different series are governed by different laws, the method of obtaining the sum of one class will not apply universally. A great variety of useful series may, however, be summed by the help of the following principles.

Since $\dfrac{pq}{n.(n+p)} = \dfrac{q}{n} - \dfrac{q}{n+p}$, $\therefore \dfrac{q}{n.(n+p)} = \dfrac{1}{p}\left(\dfrac{q}{n} - \dfrac{q}{n+p}\right)$

hence, if we have a series of fractions of the form

$$\frac{q}{n\,(n+p)},$$

their sum will be the pth part of the difference between two series of the form $\dfrac{q}{n}$ and $\dfrac{q}{n+p}$.

EXAMPLES.

Ex. 1. What is the sum of the series $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}$ ad infinitum?

Here $q=1, p=1,$

and $$S=\left\{\begin{array}{l}1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\\ \quad-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}\end{array}\right\}=1.$$

If the sum of n terms of the above series were required, we would have the nth term by $\frac{1}{n.(n+1)}$.

$$S=\left.\begin{array}{l}1+\frac{1}{2}-\frac{1}{3}\cdots\cdots\\ \quad-\frac{1}{2}-\frac{1}{3}\cdots\cdots\frac{1}{(n+1)}\end{array}\right\}=\frac{n-1}{n+1}.$$

$$=1-\frac{1}{n+1}=\frac{n}{n+1}.$$

Ex. 2. What is the sum of the series

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}, \text{ \&c., to infinity?}$$

Here $p=2,$

and $$S=\frac{1}{2}\left\{\begin{array}{l}1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\text{ \&c.}\\ \quad-\frac{1}{3}-\frac{1}{5}-\frac{1}{7}-\text{ \&c.}\end{array}\right\}=\frac{1}{2}.$$

Ex. 3. Required the sum to n terms.

Ans. $\frac{n}{2n+1}$.

Ex. 4. What is the sum of the series $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+$ &c., to infinity, and likewise to n terms?

Ans. Sum to inf. $=\frac{11}{18}$.

To n terms $=\frac{n}{3n+3}+\frac{n}{6n+12}+\frac{n}{9n+27}$.

Ex. 5. Required the sum of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c., to infinity. *Ans.* 2.

Ex. 6. What is the sum of the series $\frac{2}{3.5}-\frac{3}{5.7}+\frac{4}{7.9}-\frac{5}{9.11}$, &c., to infinity?* *Ans.* $\frac{1}{12}$.

Ex. 7. Required the sum of the last series to n terms.

Ans. $\frac{n}{12n+18}$ when n is even, $\frac{n+3}{12n+18}$ when n is odd.

Ex. 8. Required the sum of the series $\frac{1}{3.8}+\frac{1}{6.12}+\frac{1}{9.16}+\frac{1}{12.20}+$ &c., to infinity and also to n terms.

Ans. Inf. $\frac{1}{12}$,

n terms $\frac{n}{12(n+1)}$.

Ex. 9. What is the sum of the series $\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+$ &c., ad infinitum, and also to n terms?

Ans. Ad inf. $\frac{3}{4}$; to n terms $\frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)}$.

Ex. 10. Required the sum of the series $\frac{1}{1.3}-\frac{1}{2.4}+\frac{1}{3.5}-$ &c., ad infinitum, and also to n terms.

Ans. Ad inf. $\frac{1}{4}$, to n terms $\frac{1}{4}\mp\frac{1}{2(n+1)(n+2)}$, the upper sign being used when n is even.

Ex. 11. What is the sum of the series $\frac{4}{1.5}+\frac{4}{5.9}+\frac{4}{9.13}+$ &c., ad infinitum? *Ans.* 1.

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* $\frac{1}{2}=\frac{1}{1+1}=1-1+1-1+1$, ad infinitum.

200. If we have a series consisting of terms of the form

$$\frac{q}{n.(n+p)\,(n+2p)\ldots\ldots(n+mp)},$$

they may be summed by a process precisely analogous to that employed in last article; for we evidently have

$$\frac{q}{n(n+p)(n+2p)\ldots(n+mp)}=\frac{1}{mp}\left\{\frac{q}{n(n+p)\ldots(n+(m-1)p)}\right.$$
$$\left.-\frac{q}{(n+p)(n+2p)\ldots(n-mp)}\right\}.$$

EXAMPLES.

Ex. 1. Required the value of $\frac{4}{1.2.3}+\frac{5}{2.3.4}+\frac{6}{3.4.5}$, to infinity.

$$\text{Here S}=\frac{1}{2}\left\{\begin{array}{l}\frac{4}{1.2}+\frac{5}{2.3}+\frac{6}{3.4}+\&c.\\ \quad-\frac{4}{2.3}-\frac{5}{3.4}-\&c.\end{array}\right\}$$

$$=\frac{1}{2}\left\{\frac{4}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\&c.\right\}=\frac{5}{4},\ \text{(Ex. 1, Art. 199.)}$$

Ex. 2. Required the value of $\frac{1}{1.3.5}+\frac{4}{3.5.7}+\frac{7}{5.7.9}+$ &c., to infinity.

Ans. $\frac{5}{24}$.

Ex. 3. What is the sum of the series $\frac{3}{5.8.11}+\frac{9}{8.11.14}+\frac{15}{11.14.17}$ to infinity?

Ans. $\frac{13}{240}$.

Ex. 4. Required the value of $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+$ &c., to infinity.

Ans. $\frac{1}{18}$.

Ex. 5. Required the value of $\frac{1}{1.3.5.7}+\frac{2}{3.5.7.9}+\frac{3}{5.7.9.11}$ to infinity.

Ans. $\frac{1}{72}$.

Ex. 6. Required the sum of $\frac{6^2}{1.2.3.4}+\frac{7^2}{2.3.4.5}+\frac{8^2}{3.4.5.6}$ to infinity.

Ans. $\frac{89}{36}$.

CHAPTER X.

LOGARITHMS AND EXPONENTIAL EQUATIONS.

SECTION I.

Logarithms.

201. EVERY number may be considered as the power of a given root, the index of which power is called its *logarithm.*

Thus, a being supposed to be the fixed root, if $a^x=b$, $a^y=c$, x is the logarithm of b, and y of c.

202. The fixed root is called the base of the system of logarithms: and we can therefore have an infinite number of such systems. In practice the base is assumed 10, all our tables of logarithms being constructed upon this assumption.

If, therefore, we assume $a=10$, we have

$$10^0=1,\ 10^2=100,\ 10^3=1000, \&c.$$
$$10^{-1}=.1,\ 10^{-2}=.01,\ 10^{-3}=.001, \&c.$$

So that the log of $1=0$, $\log 10=1$, $\log 100=2$, &c.

$\log .1=-1$, $\log .01=-2$, $\log .001=-3$, &c.

The logarithm of any number between 1 and 10 must, therefore, be between 0 and 1. Of any number between 10 and 100, the logarithm must be between 1 and 2, and so on.

203. Let m and n be any two numbers, whose logarithms are x and y, so that we have

$$a^x = m \text{ and } a^y = n.$$

Multiplying $\quad a^{x+y} = mn,$

also $\quad a^{x-y} = \frac{m}{n}.$

But $x + y$ and $x - y$ are the logarithms of mn and $\frac{m}{n}$. Hence *the sum of the logarithms of two numbers is the logarithm of the product; and the difference of the logarithms is the logarithm of the quotient.*

Cor. n times the logarithm of a number is the logarithm of the nth power of that number.

Also the nth part of the logarithm of a number is the logarithm of the nth root of that number.

204. Let $a^x = y$, to find the value of y in a series of ascending powers of x.

Now when $x = 0$, $y = 1$, the first term of the series must therefore be 1.

Assume $\quad y = a^x = 1 + Ax + Bx^2 + Cx^3 + \&c.$

Now since A, B, C, &c., are independent of x, we shall have

$$y' = a^v = 1 + Av + Bv^2 + Cv^3 + \&c.$$

Also $a^{x+v} = 1 + A(x+v) + B(x+v)^2 + C(x+v)^3 + \&c.$

Now this last equation is evidently equal to the product of the two former. Since $a^{x+v} = a^x \times a^v$. We therefore have by multiplying and developing the powers of $x + v$ in the last,

$$\begin{array}{rrrrrl}
1 + Ax + & Bx^2 + & Cx^3 + & Dx^4 + \&c. \\
+ Av + & 2Bxv + & 3Cx^2v + & 4Dx^3v + \&c. \\
+ & Bv^2 + & 3Cxv^2 + & 6Dx^2v^2 + \&c. \\
& + & Cv^3 + & 4Dxv^3 + \&c. \\
& & + & Dv^4 + \&c. \\
= 1 + Ax + & Bx^2 + & Cx^3 + & Dx^4 + \&c. \\
+ Av + & A^2xv + & ABx^2v + & ACx^3v + \&c. \\
+ & Bv^2 + & ABxv^2 + & B^2x^2v^2 + \&c. \\
& + & Cv^3 + & ACxv^3 + \&c. \\
& & + & Dv^4 + \&c.
\end{array}$$

If, now, we cancel the terms common to both series, we shall have

$$\begin{aligned} &2Bxv + 3Cx^2v + 4Dx^3v + \&c. \\ &\quad + 3Cxv^2 + 6Dx^2v^2 \\ &\quad\quad + 4Dxv^3 \\ &= A^2xv + ABx^2v + ACx^3v \\ &\quad + ABxv^2 + B^2x^2v^2 \\ &\quad\quad + ACxv^3. \end{aligned}$$

Now since x and v are entirely independent of each other, we must have, by the principle of indeterminate coefficients, the first lines of the above expressions equal, and therefore

$$2B = A^2 \quad \text{or } B = \frac{A^2}{2}$$

$$3C = AB \quad \text{or } C = \frac{AB}{3} = \frac{A^3}{2.3}$$

$$4D = AC \quad \text{or } D = \frac{AC}{4} = \frac{A^4}{2.3.4}$$

$$\&c. \qquad = \&c.$$

Whence $y = a^x = 1 + Ax + \frac{A^2}{2}x^2 + \frac{A^3}{2.3}x^3 + \frac{A^4}{2.3.4}x^4 + \&c.$

It only remains to determine A. For this purpose let

$$x = \frac{1}{A}.$$

Then $a^{\frac{1}{A}} = 1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \frac{1}{2.3.4.5} + \&c. = e.$

$\therefore \ \frac{1}{A}\log a = \log e$, and $A = \frac{\log a}{\log e}$.

If a be the base of the system, we will have

$$A = \frac{1}{\log e}.$$

Finally if $a = e$, $A = 1$, and

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \&c. \quad \text{(A)}$$

The above is called the *Exponential Theorem.*

If e be the base, $A = \log a$,

The logarithms to the base e are called *Hyperbolic* or *Naperian* logarithms.

We may also determine A by the following process.

Assume $a = 1 + b$, so that

$$y=(1+b)^x=1+xb+x.\frac{x-1}{2}b^2+x.\frac{x-1}{2}.\frac{x-2}{3}b^3+ \&c.,$$

$$=1+(b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+\&c.)x+Bx^2+Cx^3+ \&c.$$

Consequently $A=b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+ \&c.$

$$=(a-1)-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\frac{(a-1)^4}{4}+ \&c.$$

Comparing the above with the former value of A, we obtain

$$\frac{\log a}{\log e}=\frac{\log\ (1+b)}{\log e}=b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+\&c.,$$

whence $\log (1 + b) = \log\ e\ (b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+\&c.)$ (B)

This is the logarithmic theorem.

The above, which is taken with some modifications from "Traité du Calc. Diff. et du Calc. Int." by Lacroix, though not so short as some other demonstrations, is characterized by its great elegance, and perfect rigour.

205. If we make $b = -b$ in the series (B), we have

$$\log\ (1-b)=\log\ e\ (-b-\frac{b^2}{2}-\frac{b^3}{3}-\frac{b^4}{4}-\&c.) \quad (B^1)$$

whence $\log\ (1+b)-\log\ (1-b)=\log\frac{1+b}{1-b}$

$$=2\log e\ (b+\frac{b^3}{3}+\frac{b^5}{5}+\&c.) \quad (C)$$

If now $\frac{1+b}{1-b}=n$, we have $b=\frac{n-1}{n+1}$,

$$\therefore \log n=2\log e\left\{\frac{n-1}{n+1}+\frac{1}{3}\left(\frac{n-1}{n+1}\right)^3+\frac{1}{5}\left(\frac{n-1}{n+1}\right)^5+\&c.\right\} \quad (D)$$

If now $n=2$, $\frac{n-1}{n+1}=\frac{1}{3}$, and $\log 2 = 2 \log e \left(\frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5.3^5} + \frac{1}{7.3^7} + \text{\&c.}\right)$

Calculating the value of the above series to 9 terms, we find the logarithm of

$$2 = 2.\ \log e \times .346573589,$$
$$= \log e \times .693147178,$$

and $\log 2^3 = \log 8 = 3 \log 2 = \log e \times 2.079441534.$

Again, let $n = \frac{5}{4}$, then $\frac{n-1}{n+1} = \frac{1}{9}$,

$$\therefore \quad \log \frac{5}{4} = 2 \log e. \left\{\frac{1}{9} + \frac{1}{3.9^3} + \frac{1}{5.9^5} + \text{\&c.}\right\}$$

$$\therefore \quad \log \frac{5}{4} = \log e \times .223143550,$$

and $\log 10 = \log\left(\frac{5}{4} \times 8\right) = \log e \{2.079441534 + .223143550\}$

$$= \log e \times 2.302585084,$$

but $\log 10 = 1.$

$$\therefore \quad \log e = \frac{1}{2.302585084} = .434294483 = M.$$

The quantity M is the modulus of the system. Its value to 20 places is

$$.43429,44819,03251,82765.$$

206. The series (D) converges slowly except n be small. In the actual computation of logarithms, it is important to obtain the result by using very converging series. The following are some of those frequently employed.

If in (B) and (B′) we make $b = \frac{1}{p}$, they become

$$\log \frac{p+1}{p} = M \left(\frac{1}{p} - \frac{1}{2.p^2} + \frac{1}{3p^3} - \frac{1}{4p^4} + \frac{1}{5p^5} - \text{\&c.,}\right)$$

$$\log \frac{p-1}{p} = M \left(-\frac{1}{p} - \frac{1}{2.p^2} - \frac{1}{3.p^3} - \frac{1}{4.p^4} - \frac{1}{5.p^5} - \text{\&c.,}\right)$$

and since $\log \frac{p+1}{p} = \log (p+1) - \log p$, these become

$$\log (p+1) = \log p + M \left(\frac{1}{p} - \frac{1}{2p^2} + \frac{1}{3p^3} - \text{\&c.}\right) \quad \text{(E)}$$

and $\log(p-1)=\log p-M\left(\frac{1}{p}+\frac{1}{2p^2}+\frac{1}{3p^3}+\&c.\right)$ (F)

These series enable us to find the logarithm of a number when we know that of the next greater or next less, and they converge the more rapidly as p becomes greater.

If $p=10$. we find

$$\log 11=1+M\left(\frac{1}{10}-\frac{1}{2.10^2}+\frac{1}{3.10^3}-\&c.\right)$$

$$\log 9=1-M\left(\frac{1}{10}+\frac{1}{2.10^2}+\frac{1}{3.10^3}+\&c.\right)$$

Adding (E) and (F), and transposing, we have

$$\log(p+1)=2\log p-\log(p-1)-M\left(\frac{1}{p^2}+\frac{1}{2p^4}+\frac{1}{3p^6}+\&c.\right) \quad \text{(G)}$$

By which the logarithm of a number becomes known, when that of the two preceding ones are known.

207. The following converge still more rapidly.

Put $\frac{p+1}{p}$ for n, then $\frac{n-1}{n+1}=\frac{1}{2p+1}$ and eq. D becomes

$$\log\frac{p+1}{p}=2M\left(\frac{1}{2p+1}+\frac{1}{3.(2p+1)^3}+\frac{1}{5.(2p+1)^5}+\&c.\right)$$

or $$\log(p+1)=\log p+2M\left\{\frac{1}{2p+1}+\frac{1}{3.(2p+1)^3}+\frac{1}{5.(2p+1)^5}+\&c.\right\} \quad \text{(H)}$$

Put $p+1=q^2$ then $p=q^2-1=(q+1)(q-1)$, and H becomes

$$\log q^2=\log(q+1)+\log(q-1)+2M\left(\frac{1}{2q^2-1}+\frac{1}{3(2q^2-1)^3}+\frac{1}{5(2q^2-1)^5}+\&c.\right)$$

whence

$$\log(q+1)=2\log q-\log(q-1)-2M\left(\frac{1}{2q^2-1}+\frac{1}{3(2q^2-1)^3}+\&c.\right) \quad \text{(I)}$$

which converges very rapidly.

208. The preceding are a few of the formulæ which have been invented for facilitating the computation of logarithms. They serve to show the nature of the process, and this appears to be all that is necessary, as they have little application to the general principles of mathematics. Those which are most referred to are collected below.

$$a^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \frac{A^4x^4}{1.2.3.4} + \&c. \qquad (1)$$

in which $A = \frac{\log a}{\log e}$.

e being determined by the formula

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \&c. \qquad (2)$$

Its true value to 20 places is

2.71828,18284,59045,23536.

The Exponential Theorem

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \&c. \qquad (3)$$

The Logarithmic Theorem

$$\log(1+b) = M\left(b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \&c.\right) \qquad (4)$$

If e be the base, or the logarithms be hyperbolic, we have

$$\log(1+b) = b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \&c. \qquad (5)$$

and $$\log\frac{1+b}{1-b} = 2\left(b + \frac{b^3}{3} + \frac{b^5}{5} + \&c.\right) \qquad (6)$$

SECTION II.

Exponential Equations.

209. Equations of the form $a^x = b$, $x^x = b$, &c., in which the unknown is x, are called *exponential* equations.

Those of the form $a^x = b$ may readily be solved by logarithms: thus,

Taking the logarithms, we have

$$x \log a = \log b$$

whence $$x = \frac{\log b}{\log a}.$$

210. Equations of the form $x^x = b$ may be solved by approximation, as follows.

First, find by trial two values, one greater and the other less than x, and differing from it but little. Substitute these values in the equation

$$x \log x = \log b$$

and note the results. Then as the difference of the results is to the difference between either result and $\log b$, so is the difference of the assumed numbers to a fourth term, which, applied to the assumed number corresponding to the last-mentioned result, will give the true value nearly. With this value, and the nearest of the former assumed numbers, provided a nearer number cannot be found, proceed as before, and a number will be found differing still less from the true value.

Thus let

Ex. 1. $$x^x = 20$$

Here $$x \log x = 1.301030.$$

and the value of x is readily found to lie between 2.5 and 3. Substituting these, we have

$$\begin{array}{rll} 2.5. \log 2.5 = & .9948500 & 1.4313639 \\ 3 \log 3 = & 1.4313639 & 1.3010300 \\ \hline & .4365139 & .1303339 \end{array}$$

And as $.4365139 : .1303339 :: .5 : .149$,

$\therefore \quad 3 - .149 = 2.851$ is the value of x nearly.

Ex. 2. Let $x^x = 5$, to find x. *Ans.* 2.1293.

Ex. 3. Let $x^x = 2000$, to find the value of x.
Ans. 4.8278226.

CHAPTER XI.

INTEREST AND ANNUITIES.

211. Let P represent the principal, R the rate per cent., t the time in years, I the interest, and A the amount. Then we will have $\frac{R}{100} = r$, for the interest of 1 dollar for 1 year. Consequently the interest for 1 year is Pr.

$$Prt = I,$$

and $$P + Prt = P(1 + rt) = A.$$

As, however, the student has become familiar with all the rules of Simple Interest, while studying arithmetic, it is unnecessary to develope the matter any further here, we shall therefore proceed at once to

COMPOUND INTEREST.

212. When the interest as it becomes due is taken to augment the principal, on which the interest for the next period is to be calculated, then the whole increase of the debt is called *Compound Interest.*

213. An *Annuity* is a yearly income.

214. The *present value* of any annuity is the sum which being put to compound interest will pay the amount of the annuity at the time it becomes due.

215. Since r is the interest of $1 for 1 year, $1 + r$ must be the amount of $1 for the same time.

If, then, P dollars be put out on compound interest, it will amount in one year to

$$P \cdot (1 + r).$$

Now this being the principal for the next year, the amount will be

$$P \cdot (1+r)(1+r) = P(1+r)^2.$$

Pursuing this investigation we shall find that the amount at the end of t years is $P(1+r)^t$,

Or $$A = P(1+r)^t.$$

$$\therefore \quad \log A = \log P + t \log(1+r), \quad (1)$$

$$t = \frac{\log A - \log P}{\log(1+r)}, \quad (2)$$

$$\log P = \log A - t \log(1+r). \quad (3)$$

216. If the interest be payable half yearly or quarterly, r must be taken for the interest of $1 for the half or quarter of a year, and t the number of the periods in the given time.

Examples.

Ex. 1. What is the amount of 50 dollars for 20 years, at 6 per cent. compound interest.

Here $P = 50$, $r = .06$, and $t = 20$.

Consequently we have

$\log(1+r) =$	.0253059
	20
$t \log(1+r)$	.5061180
$\log P$	1.6989700
$\log A$	2.2050880

$\therefore$ A = $160.36.

Ex. 2. What sum will amount to $1000 in 30 years, at 6 per cent. compound interest.

Here	$\log(1+r) = 1.06 =$	.0253059
	$t =$	30
		.7591770
	$\log A$ 1000 $=$	3.0000000
	$\log P$	2.2408230

$\therefore$ P = $174.11.

Ex. 3. In what time will $100, or any other sum, double itself at 6 per cent. compound interest?

Here $A = 200$,

and $$t = \frac{\log A - \log P}{\log (1+r)}.$$

$$\begin{array}{rll} \log A = & 1.3010300 & \\ \log P = & 1.0000000 & \\ \log A - \log P \quad & .3010300 \; . \; . \log - & 1.4786098 \\ \log (1+r) \quad & .0253059 \quad \log - & 2.4032218 \\ & \text{lot } t \; 11.89 \text{ years} & 1.0753880. \end{array}$$

217. In finding the amount of an annuity for a given number of years, we must recollect that the first payment will be at interest for $t-1$ years, the second for $t-2$ years, and so on. The whole amount will therefore be, the annuity being a,

$$A = a\left((1+r)^{t-1}+(1+r)^{t-2} \ldots . 1\right)$$
$$= \frac{a\{(1+r)^t-1\}}{r}. \quad \text{(Art. 63.)} \qquad (4)$$

If p be the present worth of the annuity, we must have the amount of p dollars for t years, equal to A.

Or $$p(1+r)^t = \frac{a.\{(1+r)^t-1)\}}{r}.$$

Whence $$p = \frac{a\left\{1 - \frac{1}{(1+r)^t}\right\}}{r}. \qquad (5)$$

If t be infinite $\frac{1}{(1+r)^t} = 0$, in which case $p = \frac{a}{r}$, but this is evidently the sum which will produce the annual interest a.

EXAMPLES.

Ex. 1. In what time will $500 amount to $900, at 5 per cent. compound interest? *Ans.* 12.04 years.

Ex. 2. What is the amount of an annuity of $300, foreborne for 16 years, at 4½ per cent. compound interest?
Ans. $6815.807.

Ex. 3. What sum will yield an annual income of \$630, interest being reckoned at $4\frac{1}{2}$ per cent.? *Ans.* \$14000.

Ex. 4. What is the amount of an annuity of \$700 per annum, which has been foreborne 12 years, interest being at 5 per cent.? *Ans.* \$11141.99.

CHAPTER XII.

INDETERMINATE ANALYSIS.

218. It has been shown (Art. 119) that whenever the number of independent equations is less than the number of unknown quantities, the question admits of an infinite number of answers. It is not unfrequently the case, however, that some conditions exist which partially limit the number of results. For instance, if from the nature of the question the solution is limited to positive integers, there is frequently only a single result that will apply.

Problems of this kind are called *indeterminate problems*. The results are generally required in positive integers.

219. If a and b be prime to each other, then will the remainders arising from dividing mb by a, be different for all values of m less than a. For let, if possible,

$$mb = ra + c$$

and

$$m'b = r'a + c,$$

m and m' being less than a.

From these equations we have

$$\frac{b}{a} = \frac{r - r'}{m - m'},$$

which is manifestly impossible, since $m - m'$ is less than a, and by hypothesis $\frac{b}{a}$ cannot be reduced to lower terms.

220. If a and b be prime to each other, the equation $ax - by = \pm 1$ is always possible. That is, positive integral values of x and y may be found, which will satisfy it.

For we will evidently have

$$x = \frac{by+1}{a}.$$

Now, since a and b are prime to each other, the remainders arising from dividing mb by a will be different, for all values of m less than a; one of these remainders must therefore be $a - 1$; so that we shall have

$$mb = pa + a - 1$$

$$\therefore \quad mb + 1 = pa + a = (p+1)\,a.$$

Consequently, if $y = m$, $x = p + 1$, and these being integers, the equation $ax - by = 1$ is possible.
Changing signs, we have $by - ax = 1$, and this is evidently possible.

221. **If a and b are prime to each other, the equation $ax - by = c$ admits of an infinite number of positive integral solutions.**

For since $\quad ax' - by' = 1$ is possible,
$\quad acx' - bcy' = c$ is always possible.
And putting $\quad cx' = x$ and $cy' = y$, this becomes

$$ax - by = c,$$

which is therefore always possible.

Let now $x = p$ $y = q$ be one solution,
then will $x = p + mb$, $y = q + ma$ satisfy the equation, whatever value is assigned to m. For we will have

$$ax = ap + mab,\ by = bq + mab,$$

$$\therefore \quad ax - by = ap - bq = c.$$

Cor. If a and b be not prime to each other, the equation $ax - by = c$ is impossible in integers, c being supposed to have no common measure with both a and b.

For if a and b have a common measure which is not divisible into c, one member of the equation will be divisible by a number which will not divide the other, which is manifestly absurd.

222. To find positive integer values of the equation

$$ax - by = c,$$

a and b being prime to each other.

Since $x = \frac{by + c}{a}$ is an integer, if the operation be actually performed, the remainder $b'y + c'$ must necessarily be divisible by a; so that we shall have

$$\frac{b'y + c'}{a}$$

equal to an integer, b' and c' being less than a. If now we take the difference between $\frac{ay}{a}$ and that multiple of $\frac{b'y + c'}{a}$ in which $pb'y$ may be nearest to ay, we shall have a remainder

$$\frac{b''y + c''}{a},$$

in which $b'' < b'$. By continuing this process, we shall finally arrive at a remainder of the form

$$\frac{y + p}{a}.$$

If we put this equal to r, we shall have

$$y + p = ar$$

and $$y = ar - p.$$

To illustrate the above, let the equation be

$$11\,x - 25\,y = 60.$$

Here $$x = \frac{25\,y + 60}{11} = 2\,y + 5 + \frac{3\,y + 5}{11} = \text{an Int.}$$

$$\therefore \quad \frac{12\,y + 20}{11} - \frac{11\,y}{11} = \frac{y + 20}{11} = \text{Int.} = p,$$

and $$y = 11\,p - 20,$$
$$x = 25\,p - 40,$$

in which p may be any number greater than 1.

Now, as this operation does not alter the denominator of the fraction, it is evident the numerator alone need be written; by this means the operation will be rendered more concise. Thus,

$$x = \frac{25\,y + 60}{11} = 2\,y + 5 + \frac{3\,y + 5}{11},$$

$$\begin{array}{l} 3y+5 \\ \hline 12y+20 \\ 11y+22 \\ \hline \frac{y-2}{11}=p,\ y=11p+2 \\ \qquad\qquad x=25p+10. \end{array}$$

the least values being 2 and 10.

Had we subtracted $11y$ instead of $11y+22$, we should have arrived at a result agreeing in form with that obtained by the other method.

Ex. 2. Given $9x+13y=2000$.

Here $x=\frac{2000-13y}{9}=222-2y+\frac{5y+2}{9}$,

$$\begin{array}{l} 5y+2 \\ \hline 10y+4 \\ 9y \\ \hline \frac{y+4}{9}=p \\ y=9p-4 \\ x=228-13p. \end{array}$$

By giving to p different values, we derive the following results, viz.:

$p=$	1	2	3	4	5	6	7	8
$x=$	215	202	189	176	163	150	137	124
$y=$	5	14	23	32	41	50	59	68

$p=$	9	10	11	12	13	14	15	16	17
$x=$	111	98	85	72	59	46	33	20	7
$y=$	77	86	95	104	113	122	131	140	149.

From these results, as well as the general values of x and y, we infer that the values of x must vary by the coefficient of y, and those of y by that of x.

Ex. 3. Required all the possible values of x and y in the equation $11x+5y=254$.

Ans. $x=19, 14, 9, 4,$
$y=9, 20, 31, 42.$

Ex. 4. A gentleman having a debt of $75.58 to pay, finds he has nothing but half dollar pieces to pay it with. The creditor having nothing but five franc pieces, how will they manage to settle the debt; the five franc piece being reckoned at 93 cents?

Ans. He will give 233 half dollars, and receive 44 five franc pieces.

Ex. 5. Given $11x + 35y = 500$, to find the values of x and y. *Ans.* $x = 20$,
$y = 8$.

Ex. 6. A drover bought steers for 35 dollars per head, and cows for $26. How many of each could he purchase for $1000? *Ans.* 10 steers and 25 cows.

Ex. 7. Given $7x + 13y = 71$, to find the values of x and y. *Ans.* Impossible.

Ex. 8. Given $17x + 33y = 831$, to find the values of x and y. *Ans.* $x = 12, 45$,
$y = 19, 2$.

223. To find the number of solutions of which the equation

$$ax + by = c$$

will admit.

Find values of x' and y', which will satisfy the equation

$$ax' - by' = 1,$$

$$\therefore \quad acx' - bcy' = c,$$

$$\text{but} \quad ax + by = c,$$

$$\therefore \quad ax + by = acx' - bcy',$$

$$\text{hence} \quad x = cx' - mb,$$

$$y = ma - cy'.$$

The number of solutions will therefore be the same as the number of values that can be assigned to m.

Now it is evident that $m < \frac{cx'}{b}$, and $m > \frac{cy'}{a}$.

The number of values of m will therefore correspond with the difference between the integral parts of the fractions

$$\frac{cx'}{b} \text{ and } \frac{cy'}{a},$$

except when $\frac{cx'}{b}$ is a whole number. In this case, since $m < \frac{cx'}{b}$, we must consider $\frac{b}{b}$ a fraction, and reject it. If, however, we intend to include 0 among the integral values, this last precaution need not be observed.

EXAMPLES.

Ex. 1. Determine the number of solutions the equation

$$11x + 5y = 254$$

admits of.

Here the least values of x' and y' in the equation

$$11x' - 5y' = 1$$

are $\qquad x' = 1,\ y' = 2.$

$$\therefore \qquad \frac{cx'}{b} = \frac{254}{5} = 50\frac{4}{5},$$

$$\frac{cy'}{a} = \frac{508}{11} = 46\frac{2}{11},$$

and $\qquad 50 - 46 = 4$ is the number of solutions.

Ex. 2. What is the number of integer values of x and y, that will satisfy the equation

$$21x + 5y = 20000? \qquad \textit{Ans. } 190.$$

Ex. 3. In how many ways can £1053 be paid in guineas and moidores; the guinea being 21*s*., and the moidore 27*s*.?

Ans. 111 ways.

Ex. 4. Required the number of integer values of x and y, that will satisfy the equation

$$17x + 13y = 6000. \qquad \textit{Ans. } 35.$$

224. To find the integer values of x, y, and z, in the equation

$$ax + by + cz = d.$$

If c be the greatest coefficient, then, since the values of x and y cannot be less than 1, the greatest value of z cannot exceed $\frac{d - a - b}{c}$.

Now, since $\qquad x = \frac{d - by - cz}{a},$

if we apply to this value the same principles that were employed in Art. 222, we shall arrive at a result of the form

$$\frac{y+mz+p}{a},$$

in which z may have all values, from 1 to $\dfrac{d-a-b}{c}$ inclusive, provided those values give positive integral values to x and y.

Examples.

Ex. 1. Given $17x+19y+21z=400$, to find the integral values of x, y, and z.

Here the limit of z is $\dfrac{400-17-19}{21}=17+$,

also $$x=\frac{400-19y-21z}{17}=23-y-z+\frac{9-2y-4z}{17}$$

$$\begin{array}{r} 9-2y-4z \\ \hline 72-16y-32z \\ -68+17y+34z \\ \hline \dfrac{y+2z+4}{17}=p. \end{array}$$

Whence $$y=17p-2z-4,$$
and $$x=-19p+z+28.$$

If, now, we give to z the several values, commencing at 1, we shall arrive at the following results, viz.:

z	1	2	3	4	5	6	11	12	13	14
y	11	9	7	5	3	1	8	6	4	2
x	10	11	12	13	14	15	1	2	3	4.

The remaining values of z make the results impossible.

Ex. 2. Given $12x+17y+19z=6100$, to find the values of x, y, and z.

Ans. $z=1\begin{cases} y=9,\ 21,\ 33 \ldots\ldots 224 \\ x=494,\ 477,\ 460 \ldots\ldots 1\ . \end{cases}$ 30 values.

$z=2\begin{cases} y=10,\ 22, \quad 346 \\ x=491,\ 474 \ldots\ldots\ 15 \end{cases}$ 29 values.

$z=3\begin{cases} y=11,\ 23 \ldots\ldots 347 \\ x=488,\ 471 \ldots\ldots\ 12 \end{cases}$ 29 values.

By thus proceeding, substituting 4, 5, &c., for z, the number of values would be found to be 4762. As this method, however, is very tedious, we shall, after appending a few more examples for exercise, proceed to explain a more concise method of determining the number of solutions.

Ex. 3. How many gallons at 12 cts., 15 cts., and 18 cts., must be mixed to compose 300 gallons at 17 cts. per gallon.*

Ans. 12 cts. 1, 2, 3, &c. . . . . to 49,
15 cts. 98, 96, 94, &c. . . . . to 2,
18 cts. 201, 202, 203, &c. . . . . to 249.

Ex. 4. Given $14x + 19y + 21z = 252$, to find the values of x, y, and z.

Ans. $x = 7, 4, 1,$
$y = 7, 7, 7,$
$z = 1, 3, 5.$

In this example the values of y are seen to be seven. It might easily have been inferred, a priori, that they must be 7, or one of its multiples; for since every term of the equation except the second is divisible by 7, this must be so likewise, but 19 and 7 are prime to each other. Therefore y must be a multiple of 7.

Ex. 5. Given $13x + 15y + 17z = 181$, to find the values of x, y, and z.

Ans. $x = 1, 8, 9,$
$y = 1, 4, 2,$
$z = 9, 1, 2.$

Ex. 6. Given $11x + 15y + 17z = 400$.

Ans. $z =$	1	2	3	4	5	6	7	8
$x =$	13.28	6.21	14.29	7.22	15	8.23	1.16	9
$y =$	16.5	20.9	13.2	17.6	10	14.3	18.7	11

$z =$	9	10	11	12	13	14	15	17	22
$x =$	2.17	10	3.18	11	4	12	5	6	1
$y =$	15.4	8	12.1	5	9	2	6	3	1

In all, twenty-five different results.

* In this case there will be two equations; by the elimination of one of the unknowns we will therefore arrive at an indeterminate containing but two unknown quantities, which may be solved by the preceding article.

225. In order to determine the number of solutions of which the equation $ax + by + cz = d$ admits, we must first render two at least of the coefficients prime to each other.

This is readily done, as may be seen by the following example.

Let $\qquad 9x + 12y + 16z = 424.$

Transposing $16z$, and dividing by 3, we have

$$3x + 4y = 141 - 5z + \frac{1-z}{3}.$$

$\therefore \quad \frac{1-z}{3}$ is a whole number,

and $\frac{3z}{3} + \frac{2-2z}{3} = \frac{z+2}{3} =$ a whole number $= v$.

Whence $\qquad z = 3v - 2.$

Consequently we have

$$3x+4y = 141-15v+10+1-v = 152-16v,$$

and $\quad 3x+4y+16v = 152.$

Since the values of x and y are not altered by the preceding transformation, the number of solutions will be the same as before.

Having prepared the equation as above, let a and b be the coefficients which are prime to each other, then we shall have

$$ax + by = d - cz,$$

in which z can have any value from 1 to $\frac{d-a-b}{c}$, so that z_1 being the greatest integer in the fraction $\frac{d-a-b}{c}$, we shall have z_1 equations

$$ax + by = d - c$$
$$ax + by = d - 2c$$
$$\vdots \qquad \vdots$$
$$ax + by = d - z_1 c.$$

But by (Art. 223,) the number of solutions of which the equation $ax + by = c$ admits, is equal to the difference between the integral parts of $\frac{cx'}{b}$ and $\frac{cy'}{a}$. The number of solutions in the above equations will therefore be,

$$\text{in}\begin{cases} ax+by=d-c, \text{diff. of the int. parts of } \dfrac{(d-c)x'}{b} \text{ and } \dfrac{(d-c)y'}{a} \\ ax+by=d-2c \quad \text{"} \quad \text{"} \quad \dfrac{(d-2c)x'}{b} \text{ and } \dfrac{(d-2c)y'}{a} \\ ax+by=d-3c \quad \text{"} \quad \text{"} \quad \dfrac{(d-3c)x'}{b} \text{ and } \dfrac{(d-3c)y'}{a} \\ \&c. \qquad\qquad\qquad\qquad \&c. \end{cases}$$

Now, to obtain the whole number of solutions we must add the numbers of which the various equations above admit. This is most conveniently done by obtaining the sum of the integral parts in the arithmetical series

$$\frac{(d-c)x'}{b}, \frac{(d-2c)x'}{b}, \frac{(d-3c)x'}{b}, \text{ to } \ldots z_1 \text{ terms,}$$

and subtracting therefrom the sum of the integral parts of the series

$$\frac{(d-c)y'}{a}, \frac{(d-2c)y'}{a}, \frac{(d-3c)y'}{b}, \text{ to } \ldots z_1 \text{ terms.}$$

To obtain the sum of the integral parts of the above series, first obtain the sum of the whole series and then deduct the sum of the fractional parts. Since these necessarily return in periods, it will only be necessary to calculate the sum of one period and multiply that sum by the number of periods, taking care to add the odd terms, should there not be an exact number of periods. These terms will evidently correspond with the leading terms in the series. It must also be remembered that $\frac{b}{b}$ is to be considered a fraction in the *first* series.

Examples.

Ex. 1. Required the number of solutions the equation

$$11x + 15y + 17z = 400$$

admits of.

Here the superior limit of z is $\dfrac{400-11-15}{17} = 22$.

Also the least values of x' and y' in the equation

$$11x' - 15y' = 1, \text{ are } x' = 11\ y' = 8.$$

The above series are therefore

$$\frac{383.11}{15}+\frac{366.11}{15}+\ldots\ldots\frac{26.11}{15},$$

and

$$\frac{383.8}{11}+\frac{366.8}{11}+\ldots\ldots\frac{26.8}{11},$$

the common differences being

$$\frac{17.11}{15}=12\frac{7}{15}, \text{ and } \frac{17.8}{11}=12\frac{4}{11},$$

and the number of terms 22.

Now the sums of the series are $\frac{4499.11}{15}=3299\frac{4}{15}$,

and $\frac{4499.8}{11}=3272.$

Again, the fractions in the first series are

$$\frac{13}{15}, \frac{6}{15}, \frac{14}{15}, \frac{7}{15}, \frac{15}{15}, \frac{8}{15}, \frac{1}{15}, \frac{9}{15}, \frac{2}{15}, \frac{10}{15}, \frac{3}{15}, \frac{11}{15}, \frac{4}{15}, \frac{12}{15}, \frac{5}{15},$$

$$\frac{13}{15}, \frac{6}{15}, \frac{14}{15}, \frac{7}{15}, \frac{15}{15}, \frac{8}{15}, \frac{1}{15},$$

whose sum is $12\frac{4}{15}$.

The fractions in the latter series are

$$\frac{6}{11}, \frac{2}{11}, \frac{9}{11}, \frac{5}{11}, \frac{1}{11}, \frac{8}{11}, \frac{4}{11}, \frac{0}{11}, \frac{7}{11}, \frac{3}{11}, \frac{10}{11},$$

this period being repeated twice, hence the sum is

$$2.5=10.$$

Hence the sum of the integral parts in the first series is

$$3299\frac{4}{15}-12\frac{4}{15}=3287,$$

and in the second

$$3272-10=3262,$$

the difference between which is 25, the number of solutions as in Ex. 6, Art. 224.

Ex. 2. Given $12x+17y+19z=6100$, to determine the number of solutions. *Ans.* 4762.

Ex. 3. Required the number of solutions the equation $17x + 21y + 30z = 3000$, admits of. *Ans.* 406.

Ex. 4. Required the number of solutions the equation $14x + 25y + 31z = 26428$, admits of. *Ans.* 32100.

Ex. 5. Given $3x + 7y + 11z = 86412$, to determine the number of solutions. *Ans.* 16158483.

226. To determine a number, which, divided by certain numbers, shall leave given remainders.

Let N be the number, d, d', d'', &c., the divisors, and r, r', r'', &c., the remainders.

Then we shall have

$$N = dq + r = d'q' + r' = d''q'' + r'',$$

$$\therefore \quad dq - d'q' = r' - r.$$

Let the least values of q and q' be found in this equation, then the least number N′ which will satisfy the first two conditions is $dq + r$, or $d'q' + r'$, q and q' having their least values. Also every other number that will satisfy these two conditions is included in the formula

$$dd'x + N'.$$

Consequently we will have

$$dd'x + N' = d''q'' + r''.$$

If in this equation we determine the least value of x, the number N″ corresponding to it will satisfy the first three conditions. Every other number that will satisfy these conditions is included in the formula

$$dd'd''x + N''.$$

We may thus proceed until we shall have obtained a number which will satisfy all the conditions.

Examples.

Ex. 1. To find a number, which, being divided by 3, 7, and 11, will leave the remainders 2, 5, and 7.

The required number must be of the forms

$$3x + 2,\ 7y + 5, \text{ and } 11z + 7,$$

$\therefore \qquad 3x + 2 = 7y + 5$, or $3x - 7y = 3$.

The least values of x and y that satisfy this equation are

$$x = 8,\ y = 3,$$

$\therefore \qquad 3x + 2 = 26$

is the least number that will fulfil the first two conditions. Also every number of the form

$$21x + 26,$$

will equally fulfil them. We must therefore have

$$21x + 26 = 11z + 7,$$

or $\qquad 11z - 21x = 19.$

Whence $\qquad x = 8,\ z = 17,$

and $\qquad 11z + 7 = 194,$

is the least number that will fulfil the conditions.

Every other answer must be of the form

$$3 \times 7 \times 11x + 194 = 231x + 194.$$

Ex. 2. Find the least number, which, being divided by 3, 4, 5, 6, and 7, shall leave the remainders 2, 3, 4, 5, and 6.

We shall solve this equation by a different process. Thus, if x represents the number, the expressions

$$\frac{x-2}{3},\ \frac{x-3}{4},\ \frac{x-4}{5},\ \frac{x-5}{6},\text{ and }\frac{x-6}{7},$$

must be whole numbers.

If, then, we make the first equal to p, we shall have

$$x = 3p + 2.$$

Substituting this in the second, we have

$$\frac{3p-1}{4},$$

an integer.

Whence $\qquad \frac{4p}{4} - \frac{3p-1}{4} = \frac{p+1}{4} = q,$

and $\qquad p = 4q - 1.$

Whence $\qquad x = 12q - 1.$

Substituting this in the 3d fraction, it becomes

$$\frac{12q-5}{5}.$$

Now since this must be an integer, we shall have

$$q = 5r,$$

and
$$x = 60r - 1.$$

Consequently $\frac{x-5}{6} = \frac{60r-6}{6} = 10r - 1,$

which introduces no new conditions. This might have been shown à priori; the fourth condition necessarily resulting from the 1st and 2d.

The fifth fraction $\frac{x-6}{7}$ becomes $\frac{60r-7}{7} = 8r - 1 + \frac{4r}{7}$.

Whence
$$r = 7s,$$

and
$$x = 420s - 1,$$

in which s may be any number, from 1 upward.

If $s = 1$, $x = 419$, the least number that will answer the conditions.

Ex. 3. Required the least number, which, divided by 2, 3, 5, and 7, will leave remainders 1, 2, 4, and 5, but divided by 11 will leave no remainder. *Ans.* 2189.

Ex. 4. A man has some eggs, which, when counted by twos, threes, fours, fives, or sixes, still leaves one, but when counted by sevens there are none left. What is the number? *Ans.* The least number is 301.

Ex. 5. Required the year of the Christian era, in which the solar cycle was 6, the golden number 3, and the indiction 3.* *Ans.* 1845.

* The solar cycle is a period of 28 years, at the expiration of which the days of the week return to the same days of the month, provided a common centurial year has not intervened. The first year of the Christian era being the 10th of the cycle, we must add 9 to the number of the year and divide the sum by 28, the remainder will be the number of the solar cycle.

The lunar cycle is a period of 19 years, after which eclipses return in the same order. The first year of our era being the second of this period, we must add 1 to the year, and divide by 19, the remainder is the year of the lunar cycle. This is the *golden number*.

The Roman indiction is not astronomic. It is a period of 15 years; the first of the Christian era, being the fourth of the indiction. Therefore add 3, divide by 15, and the remainder is the indiction.

In the above example we must have

$$\frac{x+9-6}{28}, \quad \frac{x+1-3}{19}, \text{ and } \frac{x}{15},$$

whole numbers.

Ex 6. What is the least whole number, which, being divided by 12 and 17, shall leave the remainders 9 and 7? *Ans.* 177.

CHAPTER XIII.

DIOPHANTINE ANALYSIS.

227. In many investigations, particularly in the higher mathematics, it becomes necessary to find values which will render irrational expressions rational. The mode of doing this, where it is possible, forms the subject of the Diophantine Analysis.

The first principles of that analysis are sufficiently simple. Some of its applications, however, have presented difficulties, the solution of which has been deemed worthy the efforts of the greatest mathematicians of the last century; and it is to two of the most illustrious of these, Euler and Lagrange, that we are indebted for most that has appeared upon the subject. The former devoted much time to the subject, and produced some investigations of the more difficult problems, which may be considered as analytical gems, well worthy of study from their beauty.

It will be impossible, in an elementary work like the present, to give any thing beyond the general principles of the science; we trust, however, that nothing will be omitted that will be found necessary in preparing the pupil for the study of the higher mathematics. Those who wish to pursue the matter further, cannot do better than to study "Barlow's Theory of Numbers," an admirable synopsis of the subjects upon which it treats, or "Legendre Theorie des Nombres," where he will find the whole subject developed by a master hand.

SECTION I.

On the Resolution of Expressions of the form

$$\sqrt{ax^2+bx+c}.$$

228. Let $a=o$, and the expression becomes

$$\sqrt{bx+c}.$$

If we put this equal to p, we shall have

$$bx+c=p^2,$$

and $$x=\frac{p^2-c}{b},$$

in which any value whatever may be given to p.

EXAMPLES.

Ex. 1. Let it be required to find a number, such that if it be multiplied by 7, and the product be increased by 10, the result may be a square.

The equation to which this question leads, is evidently

$$7x+10=p^2,$$

whence $$x=\frac{p^2-10}{7},$$

in which if we assume $p=4$; $x=\frac{6}{7}$; the other values of p will produce different results.

Ex. 2. Find values that will render $\sqrt{11x-10}$ rational.

Ans. $x=1, \frac{14}{11}, \frac{19}{11}$, &c.

Ex. 3. Render the expression $\sqrt{3x+17}$ rational.

Ans. $x=\frac{8}{3}, \frac{19}{3}, \frac{32}{3}$, &c.

229. Let the expression be of the form

$$\sqrt{ax^2 + bx}.$$

Assume $\sqrt{ax^2 + bx} = px,$

then $ax^2 + bx = p^2x^2,$

and $x = \dfrac{b}{p^2 - a},$ where p may be assumed at pleasure.

If $p = \dfrac{m}{n},$ this will become $x = \dfrac{n^2b}{m^2 - n^2a}.$

EXAMPLES.

Ex. 1. To find values of x which will make $5x^2 + 3x$ a square.

Here $a = 5,\ b = 3,$

$\therefore \qquad x = \dfrac{3}{p^2 - 5}.$

In which if $p = 3,\ x = \dfrac{3}{4}.$

Ex. 2. To find a value of x that will make $7x^2 - 15x$ a square.

Here $x = \dfrac{-15}{p^2 - 7} = \dfrac{15}{7 - p^2}.$

If $p = 2,\ x = 5.$

Ex. 3. Required a value of x that will render $\sqrt{12x^2 + 7x}$ rational.

Ex. 4. To find a number, such that if its square be divided by 12, and the result added to $\frac{1}{5}$ the number itself, the result may be a square.

Ex. 5. Render $\sqrt{8x^2 - 17x}$ rational.

230. Let the expression be of the form

$$\sqrt{a^2x^2 + bx + c},$$

in which the first term is a square.

Assume $\sqrt{a^2x^2 + bx + c} = ax + p.$

Then $a^2x^2 + bx + c = a^2x^2 + 2\,apx + p^2,$

whence $x = \dfrac{p^2 - c}{b - 2\,ap}.$

If $p = \dfrac{m}{n}$, then $x = \dfrac{m^2 - cn^2}{bn^2 - 2\,amn}.$

Ex. 1. Find such a value of x as will make $(x+3)(x-4)$ a square.

Here $(x + 3)\,(x - 4) = x^2 - x - 12,$

consequently $x = \dfrac{p^2 + 12}{-1 - 2p} =$ (if $p = -1$) 13.

Ex. 2. Find a value of x which will render $\sqrt{4x^2+17x+8}$ rational.

Ans. 8, $\frac{1}{5}$, &c.

Ex. 3. Find a value of x which will render $\sqrt{9x^2-27x+2}$ rational.

Ans. $x = 7\frac{2}{3}$.

Ex. 4. Render $\sqrt{16\,x^2 - 35\,x - 7}$ rational.

Ans.

231. Let c be a square, or the formula be of the form

$$\sqrt{ax^2 + bx + c^2}.$$

Here we may assume

$$\sqrt{ax^2 + bx + c^2} = px + c,$$

whence $ax^2 + bx + c^2 = p^2x^2 + 2\,pcx + c^2,$

and $x = \dfrac{2\,pc - b}{a - p^2}.$

Ex. 1. Find a value of x that will render $\sqrt{2x^2-27x+9}$ rational.

Ans. 3, $\frac{1}{23}$, &c.

Ex. 2. Render $\sqrt{16 - 35\,x - 7\,x^2}$ rational.

Ans.

Ex. 3. Render $\sqrt{8x^2 + 17x + 4}$ rational.

Ans.

232. General solution.

Let m and n be the roots of the equation

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

so that we shall have

$$ax^2 + bx + c = a(x - m)(x - n).$$

Assume $\sqrt{ax^2 + bx + c} = p(x - n)$,

and we obtain

$$ax^2 + bx + c = p^2(x - n)^2,$$

$\therefore$ $a(x - m) = p^2(x - n)$,

and $x = \dfrac{am - p^2 n}{a - p^2}$.

Now the roots of

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

are $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

In order to render these expressions rational, $b^2 - 4ac$ must be a square. If we put it equal to d^2, we will have

$$m = \frac{d - b}{2a} \text{ and } n = \frac{-b - d}{2a}.$$

233. We may separate the expression

$$ax^2 + bx + c$$

into two factors, when $b^2 - 4ac$ is a square, by the following simple process.

Assume $ax^2 + bx + c = (px + q)(fx + g)$

$$= pfx^2 + (pg + fq)x + gq,$$

consequently $pf = a$, $pg + fq = b$ and $gq = c$.

Squaring the second, subtracting four times the product of the first and third, and extracting the square root, we obtain

$$pq - fg = \sqrt{b^2 - 4ac} = d$$

$$\therefore \qquad g = \frac{b+d}{2p},$$

and $$q = \frac{b-d}{2f}.$$

EXAMPLES.

Ex. 1. Required a value of x which shall render

$$\sqrt{15x^2 + 22x + 7}$$

rational.

Here the factors of 15 are 5 and 3; let therefore

$$p = 5, f = 3.$$

Also $$d = \sqrt{b^2 - 4ac} = 8.$$

Hence $$g = \frac{b+d}{2p} = \frac{22+8}{10} = 3,$$

and $$q = \frac{b-d}{2f} = \frac{22-8}{6} = \frac{7}{3}.$$

$$\therefore \qquad 15x^2 + 22x + 7 = \left(5x + \frac{7}{3}\right)(3x + 3).$$

Let now $$5x + \frac{7}{3} = \frac{m^2}{n^2}(3x+3), \qquad (1)$$

and we shall have

$$\sqrt{15x^2 + 22x + 7} = \frac{m}{n}(3x+3),$$

a rational quantity.

But, from (1) we have

$$15n^2x + 7n^2 = 9m^2x + 9m^2,$$

$$\therefore \qquad x = \frac{9m^2 - 7n^2}{15n^2 - 9m^2},$$

in which m and n may be taken of any values that will render numerator and denominator of the same sign.

If $m = 1$, and $n = 1$, $x = \frac{1}{3}$.

If $m = 5$, and $n = 4$, $x = \frac{113}{15}$,

If $m = 6$, and $n = 5$, $x = \frac{149}{51}$,

&c.

234. It sometimes happens that an expression of the form

$$ax^2 + bx + c$$

may be separated into two parts, one of which is a square, and the other the product of two factors, so that we will have

$$ax^2 + bx + c = (mx + n)^2 + (px + f)(qx + g).$$

In such cases the value of x may be found by the following process:

Put $\sqrt{ax^2 + bx + c} = mx + n + d(qx + g)$.

Consequently

$$(px + f)(qx + g) = 2d(mx + n)(qx + g) + d^2(qx + g)^2.$$

$\therefore \quad px + f = 2d(mx + n) + d^2(qx + g),$

that is $px + f = 2dmx + 2dn + d^2qx + d^2g,$

and $\qquad x = \dfrac{2dn + d^2g - f}{p - 2dm - d^2q} = \dfrac{d(2n + dg) - f}{p - d(2m + dq)}.$

Examples.

Ex. 1. Find such a value of x as will make

$$\sqrt{7x^2 + 19x + 10}$$

rational.

Here we have

$$7x^2 + 19x + 10 = 4x^2 + 8x + 4 + 3x^2 + 11x + 6$$
$$= (2x + 2)^2 + (3x + 2)(x + 3),$$

$\therefore m = 2,\ n = 2,\ p = 3,\ f = 2,\ q = 1,\ g = 3,$

$\therefore$ we shall have $\quad x = \dfrac{d(4 + 3d) - 2}{3 - d(4 + d)}.$

If $d = -2$, $x = \dfrac{2}{7}$, and any other value may be given to d, which will not make x negative.

Ex. 2. Find such a value of x as shall make

$$2x^2 + 8x + 7$$

a square. *Ans.* $x = 3$.

Ex. 3. Find a value of x, which will make

$$15x^2 + 13x + 6$$

a square.

235. The above are the only cases of which a general solution has been given. Many expressions, however, occur

which do not fall under either of these forms, and yet admit of rational values; to determine these it is necessary in every case to find by trial one value; we may then determine others by the following method.

Let $x = r$ be one value that will render the expression

$$\sqrt{ax^2 + bx + c}$$

rational, and equal to p.

By (Art. 143,) transform the equation

$$ax^2 + bx + c = 0$$

into one, whose roots shall be $x - r$. The transformed equation will be of the form

$$ay^2 + b'y + p^2 = 0,$$

and if we find a value r', which will make this polynomial a square, we shall have a value

$$x = r + r',$$

which will in like manner render the original expression rational.

EXAMPLES.

Ex. 1. Required values of x which will make

$$\sqrt{7x^2 + 5x + 11}$$

rational.

A few trials will determine one value to be 2; put, then,

$$x = y + 2,$$

and we shall have

$$7x^2 + 5x + 11 = 7y^2 + 33y + 49.$$

Assume $7y^2 + 33y + 49 = (py + 7)^2 = p^2y^2 + 14py + 49,$

and $\qquad 7y + 33 = p^2y + 14p.$

Whence $\qquad y = \frac{14p - 33}{7 - p^2}.$

From the form of the above we can obtain no positive value of y. If $p = 2$, $y = -\frac{5}{3}$, and $x = \frac{1}{3}$.

And other values may be found by giving fractional values to p.

Ex. 2. Required values of x that will make

$$\sqrt{3x^2 + 7x + 6}$$

rational, $x = 1$ being one value. *Ans.*

SECTION II.

On Expressions of the Form

$$\sqrt{ax^3 + bx^2 + cx + d}.$$

236. The determination of the values of x, that will render the expression

$$\sqrt{ax^3 + bx^2 + cx + d}$$

rational, presents difficulties which have only been overcome in a general manner in two cases, viz., when the last two terms are absent, and when d is a square.

237. Let the expression be

$$\sqrt{ax^3 + bx^2}.$$

Assume $\sqrt{ax^3 + bx^2} = px,$

$\therefore$ $ax^3 + bx^2 = p^2x^2,$

and $x = \dfrac{p^2 - b}{a}.$

EXAMPLES.

Ex. 1. Find x, so that $3\,x^3 + 7\,x^2$ may be a square.

Ans. $x = 3.$

Ex. 2. Find x, so that $3\,x^3 - 5\,x^2$ may be a square.

238. If the expression be of the form

$$\sqrt{ax^3 + bx^2 + cx + d^2}.$$

Assume $ax^3 + bx^2 + cx + d^2 = (mx + d)^2$

$$= m^2x^2 + 2\,mdx + d^2.$$

If now we make $2\,md = c$, or $m = \dfrac{c}{2\,d}$,

we shall have $ax^3 + bx^2 = m^2x^2 = \dfrac{c^2}{4\,d^2}x^2.$

Whence $x = \dfrac{c^2 - 4\,bd^2}{4\,ad^2}.$

Since this contains no arbitrary quantity, it gives, of course, but one solution; others may, however, be found by the next article.

Ex. 1. Render $\sqrt{3x^3 - 5x^2 + 6x + 4}$ rational.

Ans. $x = \frac{29}{12}$.

Ex. 2. Render $\sqrt{7x^3 - 3x^2 - 4x + 16}$ rational.

Ans. $x =$

239. One value p being given, which will render the expression

$$\sqrt{ax^3 + bx^2 + cx + d}$$

rational; others may be determined as follows.

Let $ap^3 + bp^2 + cp + d = m^2$.

Transform the equation (Art. 143)

$$ax^3 + bx^2 + cx + d = 0,$$

into another whose roots shall be $x - p$.

The transformed equation will be of the form

$$a\,y^3 + b_1 y^2 + c_1 y + m^2 = 0.$$

We may by last article find a value of $y = q$ which will make this polynomial a square. Then $x = p + q$ will render the former a square.

Examples.

Ex. 1. $x=2$ renders $\sqrt{x^3 - x^2 + 2x + 1}$ rational; find another value of x that will answer.

Ans. $x = -\frac{2}{9}$.

Ex. 2. Find a value of x that will render $\sqrt{x^3 + 3}$ rational besides $x = 1$.

Ans. $x = -\frac{23}{16}$.

SECTION III.

On the Resolution of Expressions of the Form

$$\sqrt{ax^4 + bx^3 + cx^2 + dx + e}.$$

240. To find values of x that will make

$$\sqrt{a^2x^4 + bx^3 + cx^2 + dx + e}$$

rational; assume

$$a^2x^4+bx^3+cx^2+dx+e=(ax^2+mx+n)^2$$
$$=a^2x^4+2amx^3+(2an+m^2)x^2+2mnx+n^2$$

Now, if we make $2\,am = b$ or $m = \dfrac{b}{2\,a}$,

and $2\,an + m^2 = c$ or $n = \dfrac{c - m^2}{2\,a} = \dfrac{4\,a^2c - b^2}{8\,a^3}$,

we shall have

$$dx + e = 2\,mnx + n^2, \quad \text{whence} \quad x = \frac{n^2 - e}{d - 2\,mn}.$$

EXAMPLES.

Ex. 1. Required a value of x that will make $x^4 - 3\,x + 2$ a square. *Ans.* $x = \dfrac{2}{3}$.

Ex. 2. Render $4\,x^4 + 4\,x^3 + 4\,x^2 + 2\,x - 6$ rational. *Ans.* $x = 13\dfrac{1}{8}$.

241. If the expression be of the form

$$\sqrt{ax^4 + bx^3 + cx^2 + dx + e^2},$$

it may be solved by making $x = \dfrac{1}{y}$, which reduces it to the form

$$\sqrt{\left(\frac{a + by + cy^2 + dy^3 + e^2y^4}{y^4}\right)}$$

the numerator of which may be rendered rational by the formulæ of last article.

Ex. 1. Find a value of x that will make $\sqrt{2\,x^4 - 3x^3 + 1}$ rational. *Ans.* $\dfrac{3}{2}$.

Ex. 2. Find such a value of x as will make $22x^4 - 40x^3 - 40x^2 + 64x + 16$ a square. *Ans.* $\frac{8}{7}$.

242. When the first and last terms are squares, or the expression is of the form

$$\sqrt{a^2x^4 + bx^3 + cx^2 + dx + e^2},$$

we may proceed as in the preceding cases.

Or assume

$$a^2x^4 + bx^3 + cx^2 + dx + e^2 = (ax^2 + mx + e)^2$$
$$= a^2x^4 + 2\,amx^3 + (2\,ae + m^2)x^2 + 2mex + e^2.$$

This may be solved either by making the second or the fourth terms equivalent.

Thus if $2\,am = b$ or $m = \frac{b}{2\,a}$,

$$cx^2 + dx = (2\,ae + m^2)x^2 + 2\,mex,$$

whence $$x = \frac{2\,me - d}{c - (2\,ae + m^2)}.$$

If $2\,me = d$ or $m = \frac{d}{2\,e}$,

we shall have

$$bx^3 + cx^2 = 2\,amx^3 + (2\,ae + m^2)x^2$$

and $$x = \frac{(2\,ae + m^2) - c}{b - 2\,am}.$$

243. When the expression does not come under either of the preceding cases, no general solution can be given until one value has been found by trial. When such a value has been determined, the process employed in Articles 235 and 239 will generally lead to other values.

No method has been discovered for rendering rational an expression in which the unknown exceeds the fourth power —not even when one value has been found by trial.

Those who would desire to pursue this subject further, may consult "Euler's Algebra," "Barlow's Theory of Numbers," or "Legendre Théorie des Nombres," where will be found most that has been written upon this subject.

THE END.